Computing in complete local equicharacteristic Noetherian rings via topological rewriting on commutative formal power series

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Abstract

In commutative algebra, the theory of Gröbner bases enables one to compute in any finitely generated algebra over a given computable field. For non-finitely generated algebras however, other methods have to be pursued. For instance, it follows from the Cohen structure theorem that standard bases of formal power series ideals offer a similar prospect but for complete local equicharacteristic rings whose residue field is computable. Using the language of rewriting theory, one can characterise Gröbner bases in terms of confluence of the induced rewriting system. It has been shown, so far via purely algebraic tools, that an analogous characterisation holds for standard bases with a generalised notion of confluence. Subsequently, that result is utilised to prove that two generalised confluence properties, where one is actually in general strictly stronger than the other, are actually equivalent in the context of formal power series. In the present paper, we propose alternative proofs making use of tools purely from the new theory of topological rewriting to recover both the characterisation of standard bases and the equivalence between generalised confluence properties. The objective is to extend the analogy between Gröbner basis theory together with classical algebraic rewriting theory and standard basis theory with topological rewriting theory.

Keywords: standard bases, complete local rings, Cohen structure theorem, commutative formal power series, topological rewriting theory

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Introduction

In abstract algebra, one defines algebraic structures in terms of axiomatic properties of operations. No reference is made to the nature of the elements of the underlying set(s). Hence, among examples of associative commutative algebras over a field, say, for instance, the field $\mathbb{R}$ of real numbers, one might consider simple examples, such as the algebra of real numbers itself or the $\mathbb{R}$-algebra of complex numbers, but one could also look at more abstract and intricate sets of elements that can be equipped with a real algebra structure, like, for example, the $\mathbb{R}$-algebra of smooth functions on an open subset of a differentiable manifold. As mathematicians, the ability to compute, in the sense of making calculations based on the operations in an algebraic structure, is often central in obtaining new results. Even more, physicists, engineers and other applied scientists require to make effective computations to solve their problems. One big question then arises: is it enough to make approximations of the desired result, assuming it is possible to estimate how far off the approximation actually is from the answer, or do we absolutely require an exact solution of the problem? The two approaches have their own respective merit and which one is to be preferred strongly depends on the application. Approximations are the objects of study in *numerical analysis*. In this paper, we are concerned with computing exact solutions, even if they might be just symbolic expressions; this is the domain of *computer algebra* in which one implements algorithms on computers to perform effective symbolic computations so it yields an exact answer.

Just as *representation theory* offers a way of computing in even the more abstract groups by effectively looking at its elements as invertible matrices, that is, as a finite two-dimensional array of elements of a given field, associative algebras over a field $\mathbb{K}$ are always isomorphic to the quotient algebra of a *free* $\mathbb{K}$*-algebra* modulo one of its ideals. The advantage is that free algebras are combinatorial and syntactic objects that we know how to effectively represent on paper or on a computer. Here, we are concerned only with commutative algebras and, therefore, free $\mathbb{K}$-algebras take the form of the well-known polynomial algebras with coefficients in $\mathbb{K}$. We can thus take for granted: we know how to compute in polynomial algebras if we know how to calculate in $\mathbb{K}$. Now, our problem of computing in the starting associative algebra is not completely solved, as we want to work in a quotient algebra of a polynomial algebra. Thankfully, some powerful machinery to help us with that goal has been developed during the second half of the last century, namely *commutative Gröbner bases*. Their first concrete formulation as we know them today is attributed to Buchberger in his PhD thesis [1]. Gröbner bases and their generalisations actually have a multitude of very useful applications, especially in commutative algebra and algebraic geometry (see for instance [2, 3]).

The reason why polynomial algebras are of such importance is because of two things: their syntactic nature and their *universal property* among other associative commutative algebras, *i.e.* that property we mentioned about any algebra being isomorphic to a quotient of a polynomial algebra. However, consider a $\mathbb{K}$-algebra A that is not finitely generated, that is to say, there is no finite subset $\{x_1, \cdots, x_n\} \subseteq A$ that generates A as a $\mathbb{K}$-algebra, *i.e.* such that the canonical morphism $\mathbb{K}[x_1, \cdots, x_n] \to A$ is surjective. This means that for such an algebra A, we have to consider infinitely many indeterminates in our polynomial algebra and, it turns out, many important results from Gröbner basis theory do not apply anymore. However, alternative tools are available for certain classes of non-finitely generated $\mathbb{K}$-algebras. Indeed, even before Buchberger, Hironaka had developed in [4, 5] the notion of *standard basis* of an ideal of commutative formal power series when he was interested in the problem of *resolution of singularities* in algebraic geometry (for more details, see [6, 7]). As stated in Proposition 11, commutative formal power series with coefficients in $\mathbb{K}$ are not finitely generated as $\mathbb{K}$-algebras and therefore, standard bases are useful in that case, where Gröbner bases fail to help.

Now, the *Cohen structure theorem* originally proven in [8] gives us insight into a certain class of rings that can be equipped with the structure of an algebra over an appropriate field. We recall that result in Theorem 12. In that theorem, Statement 3 expresses the kind of universal property for commutative formal power series that polynomial algebras enjoy but for the specific class of *complete local equicharacteristic Noetherian rings*. Those rings actually contain a subfield $\mathbb{K}$, which turns out to be isomorphic to their so-called *residue field*, and, when seen as $\mathbb{K}$-algebras, they are isomorphic to a quotient of a commutative formal power series algebra in finitely many indeterminates. Moreover, Statement 4 enables us to prove Proposition 17 which states that even those complete local equicharacteristic Noetherian rings are not finitely generated when considered as algebras over their residue field and, therefore, Gröbner bases are of no use. Among examples of complete local equicharacteristic Noetherian rings, one can find commutative formal power series rings, and complete local Cohen-Macaulay rings to name a few.

So far, we have given motivations for the notion of standard basis by comparing it with Gröbner bases. However, unlike for Gröbner bases and so-called algebraic or linear rewriting theory, there is a lack of literature on the subject of standard bases viewed from *rewriting theory*. But first, what is rewriting theory? It is a language to formalise the concept of sequences of computations. It originally comes from group theory in order to solve decision problems and it then got adapted in computer science and, more precisely, to fulfill the needs to formalise and unify certain models of computations in computability theory as well as give explicit formal systems in logic. Rewriting theory suffers from a lack of easily accessible literature on the topic if it were not for the two standard references [9, 10]. Now, despite rewriting theory being originally motivated by its use in group theory, logic, term rewriting and lambda-calculus, it has proven to be exceptionally powerful at providing a language for the theory of Gröbner bases (see [11, 12]). In that context, a Gröbner basis is characterised by the so-called *confluence* property of the rewriting system it induces, which exactly means that the system enjoys an underlying determinism. More precisely, this confluence property states that, no matter what path of sequences of computation one chooses to reduce the polynomial, one always reaches the same result, called the *normal form of the polynomial*. Such normal forms can be used as canonical representatives of the equivalence classes in the quotient algebra. This allows to keep our computations in the polynomial algebra, where we know how to calculate, and, at each step, reducing to normal form to get the corresponding equivalence class.

As we mentioned however, standard bases had so far not been investigated from a rewriting-theoretic perspective, strictly speaking. Only recently in [13], Chenavier used algebraic tools to characterise standard bases of commutative formal power series with the notion of δ-confluence, a generalisation of the classical confluence property. The author also introduced a generalisation of the transitive reflexive closure of the one-step reduction relation which is studied in classical rewriting theory. That generalised relation, called the *topological rewriting relation*, depends on a topology on the underlying set of the rewriting system and, therefore, that new theory is called *topological rewriting theory*. It provides a framework in which one can investigate more efficiently rewriting systems which fail to be confluent or *terminating* in the classical sense. This includes formal power series for the algebraists but also, infinitary term-rewriting and infinitary lambda-calculus [14–16] for the computer scientists. However, the generalised confluence property studied in infinitary Σ/λ-term rewriting is different syntactically from the δ-confluence. We will call the former *infinitary topological confluence* property and the latter *finitary topological confluence*. From the general point of view of topological rewriting theory, it is shown in [17] that infinitary topological confluence is strictly stronger than finitary topological confluence. However, the authors used the above rewriting-theoretic characterisation of standard bases in terms of δ-confluence to show that infinitary and finitary topological confluences are actually equivalent in commutative formal power series.

The goal of the present paper consists in giving a more rewriting-theoretic approach to standard bases of commutative formal power series than before presented in the mentioned literature. We will prove by new and more elementary means the results of [13] about the characterisation of standard bases in terms of finitary topological confluence and of [17] about the equivalence between the notions of generalised confluence. This will be part of the main contribution of this article stated among the results of Theorem 49. That theorem states the properties that are analogous to the Gröbner basis theory but for standard bases. We will also prove in Theorem 48 that the equivalence relation generated by the topological rewriting relation is exactly the congruence relation modulo the ideal generated by the rewrite rules.

In Section 1, we will first introduce in Subsection 1.1 the notions of local rings, complete local rings, complete local equicharacteristic Noetherian rings, then, in Subsection 1.2 we will present monomial orders, commutative formal power series, the Cohen structure theorem and some of its consequences. In Section 2, we start by recalling the basics of topological rewriting theory, then we give new results regarding normal forms and some uniqueness properties and conclude Subsection 2.1 with the definitions of infinitary and finitary topological confluences. In Subsection 2.2, we study some notion of attractivity of normal forms formalised in the context of topological rewriting systems over a metric space. We follow by Subsection 2.3 in which we prove that, under some previously defined assumptions, the finitary topological confluence property implies the unique normal form property. Then, in Subsection 3.1, we remind the reader about the reduction on formal power series that gives rise to the topological rewriting system we want to study. In Subsection 3.2, we recall the original language in which standard bases were characterised. In Subsection 3.3, we prove a few useful lemmas and the fact that the congruence relation modulo the ideal generated by the rewrite rules is exactly the equivalence relation generated by the topological rewriting relation (with or without chains). Finally, we will conclude with the main contribution of the paper: characterisations of standard bases in terms of rewriting theory.

1 Preliminary notions

1.1 Complete local Noetherian rings

The basic object of interest for our purposes are the so-called *local rings*. Those are the basis of many modern theories on manifolds and schemes in differential and algebraic geometries, as the main object of study in those cases can be viewed as a *locally-ringed space*, that is to say, a topological space (with potentially additional properties) together with a sheaf of rings (or algebras) such that the stalk above each point is a local ring. Prime examples of those local rings as stalks are sets of germs of continuous, differentiable, smooth, analytic, holomorphic, or rational functions on the base space or topological manifold.

Definition 1 (Local ring) Let K be a commutative ring. If K admits a unique maximal ideal, then K is said to be a *local ring*, where a maximal ideal is any ideal I of K such that we have $I \neq K$ and, for every ideal J of K, if $I \subsetneq J$, then $J = K$.

If K is a local ring and $\mathfrak{m}$ is a maximal ideal of K, we write $(K, \mathfrak{m})$ to express the fact that K is a local ring with unique maximal ideal $\mathfrak{m}$. By well-known basic results of commutative algebra, if $(K, \mathfrak{m})$ is a local ring, then the quotient ring $\kappa := K/\mathfrak{m}$ is actually a field, since $\mathfrak{m}$ is maximal. We call that field κ the *residue field* of the local ring $(K, \mathfrak{m})$. The topology on rings we are concerned with in our context is the I*-adic topology* which is induced by an ideal I in the ring at study.

Definition 2 (I-adic topology) Let K be a commutative ring and I be one of its ideals. We define the I*-adic topology on* K as the topology whose basis of open subsets is given by the sets:

$$\left\{x + I^k \,\middle|\, x \in K, k \in \mathbb{N}\right\}.$$

Endowed with that I-adic topology, K is a *topological ring.*

In general, this topology is not metrisable. However, a consequence of the Artin-Rees lemma [18] known as the *Krull's Intersection Theorem* allows us to describe a metric compatible with this topology when the ring is commutative Noetherian as well as an integral domain or a local ring.

Proposition 3 *Let K be a commutative Noetherian ring which is an integral domain or a local ring. For any proper ideal I of K, the following map d_I:*

$$\begin{array}{rccl} d_I: & K \times K & \longrightarrow & \mathbb{R}_{\geq 0} \\ & (a, b) & \longmapsto & d_I(a, b) := \dfrac{1}{2^{\sup\{k \in \mathbb{N} \,|\, a - b \in I^k\}}}, \end{array}$$

with the convention that, if the set $\left\{k \in \mathbb{N} \,\middle|\, a - b \in I^k\right\}$ *is not bounded above, then the supremum is equal to* ∞ *and, subsequently,* $2^{-\infty} = 0$*, defines a metric on* K *which is compatible with the* I*-adic topology.*

It is then natural to ask about the metric space completion of an I-adic topology induced by a metric.

Definition 4 (I-adic completion) Let K be a commutative Noetherian ring and I an ideal of K in such a way that (K, d_I) is a metric space. Then, the *I-adic completion* of K is the metric space completion $(\hat{K}, \hat{d}_I)$ of (K, d_I).

Remark 5 Actually, the I-adic completion need not be defined only for topological rings which are also metric spaces but we restrict ourselves to that case here for simplicity.

Since K with the I-adic topology is a topological ring, by continuity of operations, the I-adic completion $\hat{K}$ together with the topology induced by the metric $\hat{d}_I$ is a *complete topological ring* which contains K as a subring. Let us show that the $\hat{I}$-adic topology on $\hat{K}$, where $\hat{I}$ is the ideal generated by I in $\hat{K}$ when I is viewed as a subset of $\hat{K}$, is the same as the topology induced by $\hat{d}_I$ on $\hat{K}$. First, recall that it can be shown by induction that, for any $r \in \mathbb{N}$, we have $\hat{I}^r = \hat{K}I^r$. Now, it suffices to show that, for any point x in any basic open subset U in one of the topologies, there exists a basic open subset V in the other topology such that $x \in V \subseteq U$.

In one direction, let $x \in \hat{K}$ and $\rho \in \mathbb{R}_{>0}$. Consider r the smallest integer such that $2^{-r} < \rho$. Let $y \in x + \hat{I}^r$ and take $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ two Cauchy sequences in K that represent x and y respectively in $\hat{K}$. Hence, there exists $\hat{a} \in \hat{I}^r$ such that $\lim_{k\to\infty} x_k - y_k = \hat{a}$. But, by the reminder above, it follows that there exist $\ell \in \mathbb{N}$, $\hat{z}_1, \cdots, \hat{z}_\ell \in \hat{K}$ and $a_1, \cdots, a_\ell \in I^r$ such that $\hat{a} = \sum_{i=1}^{\ell} \hat{z}_i a_i$. Consider, for each $i \in \{1, \cdots, \ell\}$, a Cauchy sequence $(z_{i,k})_{k\in\mathbb{N}}$ in K that represents $\hat{z}_i$ in $\hat{K}$. It follows that:

$$\lim_{k\to\infty} \sum_{i=1}^{\ell} z_{i,k} a_i = \lim_{k\to\infty} x_k - y_k.$$

It is clear that, for any $k \in \mathbb{N}$, the set $\left\{j \in \mathbb{N} \,\middle|\, \sum_{i=1}^{\ell} z_{i,k} a_i \in I^j\right\}$ contains r by construction. Hence, its supremum is well-defined and is greater than or equal to r.

Now, for $k \in \mathbb{N}$, we get:

$$d_I(x_k, y_k) = 2^{-\sup\left\{j\in\mathbb{N} \,\middle|\, x_k - y_k \in I^j\right\}} = d_I(x_k - y_k, 0),$$

$$d_I(x_k, y_k) \leq d_I\left(x_k - y_k, \sum_{i=1}^{\ell} z_{i,k} a_i\right) + d_I\left(\sum_{i=1}^{\ell} z_{i,k} a_i, 0\right) \qquad \text{triangular inequality,}$$

$$d_I(x_k, y_k) \leq d_I\left(x_k - y_k, \sum_{i=1}^{\ell} z_{i,k} a_i\right) + 2^{-\sup\left\{j\in\mathbb{N} \,\middle|\, \sum_{i=1}^{\ell} z_{i,k} a_i \in I^j\right\}} \qquad \text{by definition,}$$

$$d_I(x_k, y_k) \leq d_I\left(x_k - y_k, \sum_{i=1}^{\ell} z_{i,k} a_i\right) + 2^{-r} \qquad \text{see above.}$$

Finally, by making k tend to infinity, we get:

$$\begin{aligned}\hat{d}_I(x, y) = \lim_{k\to\infty} d_I(x_k, y_k) \leq \lim_{k\to\infty} d_I\left(x_k - y_k, \sum_{i=1}^{\ell} z_{i,k} a_i\right) + 2^{-r},\\ \hat{d}_I(x, y) \leq 0 + 2^{-r},\\ \hat{d}_I(x, y) < \rho.\end{aligned}$$

Conversely, let $x \in \hat{K}$ and $r \in \mathbb{N}$. Let us show that there exists $\rho \in \mathbb{R}_{>0}$ such that every $y \in \hat{K}$ satisfying $\hat{d}_I(x, y) < \rho$ is in $x + \hat{I}^r$. Consider $\rho := 2^{-r}$. Let $y \in \hat{K}$ be such that $\hat{d}_I(x, y) < \rho$. Take $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ two Cauchy sequences in K representing x and y respectively in $\hat{K}$. It follows that there exists $K_\rho \in \mathbb{N}$ such that, for all $k \geq K_\rho$, we have $\hat{d}_I(x_k, y_k) = d_I(x_k, y_k) < \rho = 2^{-r}$, hence, $\sup\left\{j \in \mathbb{N} \,\middle|\, x_k - y_k \in I^j\right\} > r$. This means that, for any $k \geq K_\rho$, $r \in \left\{j \in \mathbb{N} \,\middle|\, x_k - y_k \in I^j\right\}$ and, thus, $x_k - y_k \in I^r$. Making k tend towards infinity, we obtain $x - y \in \hat{I}^r$, thus $y \in x + \hat{I}^r$, as desired.

To understand the following definition, recall that, by Proposition 3, the $\mathfrak{m}$-adic topology on a local ring $(K, \mathfrak{m})$ is metrisable by the metric $d_\mathfrak{m}$ defined in that proposition.

Definition 6 (Complete local Noetherian ring) Let $(K, \mathfrak{m})$ be a local ring where K is Noetherian. If $(K, d_\mathfrak{m})$ is complete as a metric space, then we say that $(K, \mathfrak{m})$ is a *complete local Noetherian ring*.

As a complete local Noetherian ring is not necessarily an integral domain (and thus not a field), the characteristic of the ring can be something else than 0 or a prime number. It turns out that it can also be a power of a prime number. However, we will dismiss that possibility by considering only complete local Noetherian rings that have the same characteristic as their residue field, which is thus necessarily 0 or prime.

Definition 7 (Equicharacteristic) Let $(K, \mathfrak{m})$ be a local ring and $\kappa := K/\mathfrak{m}$ its residue field. If the characteristic of K is the same as the characteristic of κ, then $(K, \mathfrak{m})$ is said to be *equicharacteristic*.

Remark 8 For any local ring $(K, \mathfrak{m})$, being equicharacteristic is equivalent to containing a subring which is actually a field. Some authors prefer to use that other statement and might not even use the term equicharacteristic (for instance [19]).

It is a known result from [20] that, for an ideal I in a commutative ring K, the ideals of K are topologically closed in the I-adic topology if, and only if, I is contained in the so-called *Jacobson radical*, which in our context is just the intersection of all maximal ideals of K. In particular, we have the following proposition.

Proposition 9 *Any ideal in a local ring $(K, \mathfrak{m})$ where K is Noetherian is topologically closed for the $\mathfrak{m}$-adic topology.*

1.2 Formal power series and standard bases

Throughout this subsection, n is a fixed positive integer, $x_1, \cdots, x_n$ are distinct indeterminates and $\mathbb{K}$ a (commutative) field of arbitrary characteristic. Denote by $[x_1, \cdots, x_n]$ (or $[\mathbf{x}]$ for short) the free commutative monoid generated by $\{x_1, \cdots, x_n\}$ (whose law of composition is denoted multiplicatively by $\cdot$), call elements of $[\mathbf{x}]$ *monomials* and denote by 1 the *empty monomial*, *i.e.* the identity element of $[\mathbf{x}]$. Elements of $[\mathbf{x}]$ are of the form $\mathbf{x}^\mu := x_1^{\mu_1} \cdots x_n^{\mu_n}$ where $\mu := (\mu_1, \cdots, \mu_n) \in \mathbb{N}^n$. In particular, we have the identity $1 = \mathbf{x}^{0_n} = x_1^0 \cdots x_n^0$. Define the *total degree* of a monomial $m := \mathbf{x}^\mu \in [\mathbf{x}]$ by the formula $\deg(m) := |\mu| := \mu_1 + \cdots + \mu_n$.

Definition 10 (Monomial order) We call *monomial order on* $[\mathbf{x}]$ any total order $\leq$ on $[\mathbf{x}]$ such that, for all $m, m_1, m_2 \in [\mathbf{x}]$, if we have $m_1 \leq m_2$, then $m \cdot m_1 \leq m \cdot m_2$.

As any partial order, a monomial order $\leq$ is uniquely determined by its associated strict order $<$. Therefore, we will use the term "monomial order" for $\leq$ and $<$ interchangeably. Note how, for any monomial order $<$ on $[\mathbf{x}]$, the opposite order $<^{\mathrm{op}}$ is also a monomial order on $[\mathbf{x}]$. A monomial order $<$ on $[\mathbf{x}]$ is said to be *admissible* if, for all $m \in [\mathbf{x}]$, $1 \leq m$. This is equivalent to saying that $\leq$ is a well-order on $[\mathbf{x}]$. Some authors [3] use the term *global order* for admissible monomial order and call the opposite order of a global order a *local order*. We say that a monomial order $<$ is *compatible with the degree* if, for all $m_1, m_2 \in [\mathbf{x}]$ such that $\deg(m_1)$ is strictly smaller than $\deg(m_2)$, then necessarily $m_1 < m_2$. An example of an admissible monomial order compatible with the degree is the "degree-lexicographic order" (or "deglex" for short). An admissible monomial order compatible with the degree is of order type ω, where ω denotes as usual the first infinite ordinal.

Denote by $\mathbb{K}[x_1, \cdots, x_n]$ ($\mathbb{K}[\mathbf{x}]$ for short) the $\mathbb{K}$-*algebra of n-multivariate polynomials with coefficients in* $\mathbb{K}$. By construction, any polynomial $f \in \mathbb{K}[\mathbf{x}]$ is a finite linear combination of monomials in $[\mathbf{x}]$ with coefficients in $\mathbb{K}$ and, thus, it makes sense to define, for any monomial $m \in [\mathbf{x}]$, the scalar $\langle f \mid m \rangle$ called the *coefficient in f associated to m*. It follows that the set $\mathrm{supp}\,(f) := \{m \in [\mathbf{x}] \mid \langle f \mid m \rangle \neq 0\}$, called the *support of the polynomial f* is finite and that any polynomial f can thus be represented by the finite linear combination $f = \sum_{m \in \mathrm{supp}(f)} \langle f \mid m \rangle\, m$. The set $\mathbb{K}$ can be embedded into $\mathbb{K}[\mathbf{x}]$ by the morphism of algebras $\mathbb{K} \ni \lambda \mapsto \lambda 1 \in \mathbb{K}[\mathbf{x}]$ where, here, 1 is the empty monomial. The image of $\mathbb{K}$ by that morphism is exactly the set of *constant polynomials* and it is the complementary subspace in $\mathbb{K}[\mathbf{x}]$ to the subspace given by the ideal $I(x_1, \cdots, x_n)$ in $\mathbb{K}[\mathbf{x}]$ generated by $\{x_1, \cdots, x_n\}$. Denote that ideal by $I(\mathbf{x})$

for short and note how it follows that $I(\mathbf{x})$ is a proper ideal of $\mathbb{K}[\mathbf{x}]$ since 1 is a non-zero constant polynomial. By Hilbert's Basis Theorem, $\mathbb{K}[\mathbf{x}]$ is Noetherian and, since we assume $\mathbb{K}$ is a field, also an integral domain, we can conclude that the $I(\mathbf{x})$-adic topology on the ring $\mathbb{K}[\mathbf{x}]$ is metrisable. Denote the associated metric by $\delta := d_{I(\mathbf{x})}$ defined in Proposition 3.

We define the topological ring of (commutative) *formal power series in n indeterminates with coefficients in* $\mathbb{K}$ as the $I(\mathbf{x})$-adic completion (see Definition 4) of $(\mathbb{K}[\mathbf{x}], \delta)$ and we denote it $(\mathbb{K}[[x_1, \cdots, x_n]], \delta)$ (or $(\mathbb{K}[[\mathbf{x}]], \delta)$ for short). Since $\mathbb{K}$ is embedded into the ring $\mathbb{K}[\mathbf{x}]$ and $\mathbb{K}[\mathbf{x}]$ into $\mathbb{K}[[\mathbf{x}]]$, it follows that $\mathbb{K}[[\mathbf{x}]]$ is actually a topological $\mathbb{K}$-algebra by defining the scalar multiplication as the formal power series multiplication restricted to the subring identified with $\mathbb{K}$. It is a well-known result that this definition of $\mathbb{K}[[\mathbf{x}]]$ coincides with the so-called "large algebra associated to the monoid $[\mathbf{x}]$" of Bourbaki [21]: $\mathbb{K}[[\mathbf{x}]]$ is thus defined as the set of functions from $[\mathbf{x}]$ to $\mathbb{K}$, including those whose support is infinite. In other words, an element f of $\mathbb{K}[[\mathbf{x}]]$ can be represented as a, possibly infinite, linear combination of monomials in $[\mathbf{x}]$ with coefficients in $\mathbb{K}$ by formally writing it as the sum of every monomial m in $[\mathbf{x}]$ preceded by its associated coefficient, that is to say, the scalar $f(m)$:

$$f =: \sum_{m \in [\mathbf{x}]} f(m)m.$$

Extending the notations for polynomials, we set $\langle f \,|\, m \rangle := f(m)$ the coefficient associated to a monomial $m \in [\mathbf{x}]$ in a formal power series $f \in \mathbb{K}[[\mathbf{x}]]$ as given by that linear combination representation, as well as $\operatorname{supp}(f) := \{m \in [\mathbf{x}] \,|\, \langle f \,|\, m \rangle \neq 0\}$ for a formal power series $f \in \mathbb{K}[[\mathbf{x}]]$.

Now, since δ induces the $I(\mathbf{x})$-adic topology on $\mathbb{K}[[\mathbf{x}]]$ when $I(\mathbf{x})$ is viewed as an ideal of $\mathbb{K}[[\mathbf{x}]]$, we can give a somewhat more convenient definition of δ:

$$\forall f, g \in \mathbb{K}[[\mathbf{x}]], \quad \delta(f, g) = \frac{1}{2^{\operatorname{val}(f-g)}}, \tag{1}$$

where $\operatorname{val}(h)$ is equal to the smallest degree of monomials in $\operatorname{supp}(h)$, for any non-zero formal power series $h \in \mathbb{K}[[\mathbf{x}]] \setminus \{0\}$ or ∞ if $h = 0$.

Let us prove the following property.

Proposition 11 *The algebra $\mathbb{K}[[\mathbf{x}]]$ is not finitely generated as a $\mathbb{K}$-algebra.*

Proof First of all, recall that a commutative ring K is said to be *Jacobson* when every prime ideal is the intersection of maximal ideals. If $\mathbb{K}$ is a field, then it is straightforwardly Jacobson. By a general form of Hilbert's *Nullstellensatz* [22], if K is a Jacobson ring then any finitely generated K-algebra is also Jacobson. Let $\mathbb{K}$ be a field. Then any finitely generated $\mathbb{K}$-algebra K which turns out to be a local ring is of Krull dimension 0: indeed, let $\mathfrak{m}$ be the unique maximal ideal of K, then, since K is Jacobson, the only prime ideal is $\mathfrak{m}$. Now, if furthermore K is an integral domain, that is to say, $\{0\}$ is a prime ideal, then it is once again clear that the latter is the unique maximal ideal, and thus K is a field. Hence, by contrapositive, let K

be a $\mathbb{K}$-algebra which is local as a ring as well as an integral domain, then, if K is not a field, it cannot be finitely generated as a $\mathbb{K}$-algebra. Finally, it is known that $\mathbb{K}[[\mathbf{x}]]$ is a $\mathbb{K}$-algebra and is local as a ring as well as an integral domain but not a field. $\square$

Now, let us recall the *Cohen Structure Theorem*, originally proved by COHEN in [8]. For our purposes on complete local equicharacteristic Noetherian rings, we will reformulate the theorem from [19, AC IX.30, Paragraphe 4, Numéro 3, Théorème 2]:

Theorem 12 *Let $(K, \mathfrak{m})$ be a complete local equicharacteristic Noetherian ring. Let us denote by $\kappa := K/\mathfrak{m}$ its residue field and by d the Krull dimension of K.*

1. *The ring K contains a subring $\mathbb{K}$ which maps isomorphically onto κ through the natural projection $K \longrightarrow K/\mathfrak{m} = \kappa$ and, therefore, is a field of same characteristic as both κ and K. Thus, K is a $\mathbb{K}$-algebra. Such a field $\mathbb{K}$ is a called* field of representatives *for $(K, \mathfrak{m})$ (or, as originally called by* COHEN, *coefficient field).*
2. *The quotient $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over $\mathbb{K}$. Let m be its $\mathbb{K}$-dimension.*
3. *There exists an ideal I of $\mathbb{K}[[x_1, \cdots, x_m]]$ such that K and $\mathbb{K}[[x_1, \cdots, x_m]]/I$ are isomorphic as $\mathbb{K}$-algebras.*
4. *There exists a sub-$\mathbb{K}$-algebra K_0 of K such that K_0 is isomorphic to $\mathbb{K}[[x_1, \cdots, x_d]]$ and K is a finitely generated K_0-module.*
5. *If $(K, \mathfrak{m})$ is* regular, *i.e., if $d = m$, then the $\mathbb{K}$-algebras K and $\mathbb{K}[[x_1, \cdots, x_d]]$ are isomorphic.*

Let us now introduce standard bases of formal power series. In order to accomplish that, we need to define the notions of leading monomials and leading coefficients.

Definition 13 Let $<$ be an admissible monomial order on $[\mathbf{x}]$ and $f \in \mathbb{K}[[\mathbf{x}]] \setminus \{0\}$. Define the *leading monomial* of f for $<$ as follows:

$$\mathrm{lm}\,(f) := \min_{<} \mathrm{supp}\,(f) \in [\mathbf{x}].$$

We then define the *leading coefficient* of f for $<$ as $\mathrm{lc}\,(f) := \langle f \mid \mathrm{lm}\,(f) \rangle \in \mathbb{K}$.

Definition 14 (Standard basis) Let I be an ideal in $\mathbb{K}[[\mathbf{x}]]$ and $<$ be an admissible monomial order. A *standard basis of I for $<$* is any set $G \subseteq I \setminus \{0\}$ such that, for all $f \in I \setminus \{0\}$, there exist $g \in G$ and $m \in [\mathbf{x}]$ that satisfy:

$$\mathrm{lm}\,(f) = m \cdot \mathrm{lm}\,(g)\,.$$

In the commutative setting that we study here, there always exists a *finite* standard basis for any ideal of $\mathbb{K}[[\mathbf{x}]]$ and any admissible monomial order.

It is known since HIRONAKA [4, 5] that, for a fixed ideal I in $\mathbb{K}[[\mathbf{x}]]$ and a fixed monomial order $<$, for any $f \in \mathbb{K}[[\mathbf{x}]]$, there exists a unique formal power series, let us denote it here by $r_f \in \mathbb{K}[[\mathbf{x}]]$, such that $f \equiv r_f \mod I$ and no monomial in $\mathrm{supp}\,(r_f)$ is a multiple of a leading monomial for $<$ of a formal power series in I. That formal power series r_f is called the *Hironaka remainder* of f for $<$ modulo I. It will serves as a canonical representative of the equivalence class of f modulo I. The purpose of standard bases was primarily to compute effectively that Hironaka remainder. In Subsection 3.3, we will show that Hironaka remainders are actually just the normal forms of a topological rewriting system which satisfies a certain notion of confluence.

Here is a useful proposition which relates the degree of the leading monomial of the difference of two formal power series for a certain monomial order with the distance between the two formal power series.

Proposition 15 *Let $<$ be an admissible monomial order on $[\mathbf{x}]$ which is compatible with the degree. Then, for all $f, g \in \mathbb{K}[[\mathbf{x}]]$ such that $f \neq g$, we have:*

$$\delta(f,g) = \frac{1}{2^{\deg(\operatorname{lm}(f-g))}}.$$

Proof Indeed, let $h \in \mathbb{K}[[\mathbf{x}]] \setminus \{0\}$, since the monomial order $<$ is admissible, the leading monomial $\operatorname{lm}(h)$ is exactly the least (for $<$) monomial in the support $\operatorname{supp}(h)$. Now, since the order $<$ is compatible with the degree, *i.e.* the degree function on monomials is non-decreasing, it follows that $\deg(\operatorname{lm}(h))$ is minimal among all the $\deg(m)$ with $m \in \operatorname{supp}(h)$. This exactly means that $\deg(\operatorname{lm}(h)) = \operatorname{val}(h)$, hence the result. □

To end this subsection, we will prove Proposition 17. For that purpose, we will require the following lemma:

Lemma 16 (Artin-Tate [23]) *Let K be a commutative Noetherian ring and let $A_0 \subseteq A$ be two commutative K-algebras. If A is finitely generated as a K-algebra and if A is a finitely generated A_0-module, then A_0 is finitely generated as a K-algebra.*

Now, using previous results we prove the following proposition.

Proposition 17 *Let $(K, \mathfrak{m})$ be a complete local equicharacteristic Noetherian ring which has a positive Krull dimension. Let $\kappa := K/\mathfrak{m}$ be its residue field. Then, K is not finitely generated as a κ-algebra, where the action of κ is defined via Statement 1 of Theorem 12.*

Proof By Theorem 12 statement number 4, there exists a sub-κ-algebra K_0 of K isomorphic to a κ-algebra of formal power series in finitely many variables in such a way that K is actually a finitely generated K_0-module. However, by Proposition 11, K_0 cannot be finitely generated as a κ-algebra. Hence, by contrapositive of Lemma 16, we conclude that K cannot be finitely generated as a κ-algebra. □

A consequence of Proposition 17 is that, if we were to use Gröbner bases methods to compute in a complete local equicharacteristic Noetherian ring, we would have to work on a polynomial ring with infinitely many variables, for which many essential results of Gröbner bases in finitely many variables fail to translate, especially when it comes to effectiveness. This is why standard bases are a useful alternative tool.

It follows from Proposition 9 and the fact that $I(\mathbf{x})$ is the unique maximal ideal of the local ring $\mathbb{K}[[\mathbf{x}]]$ with which we defined the topology that any ideal in $\mathbb{K}[[\mathbf{x}]]$ is topologically closed for the topology induced by the metric δ.

2 Topological rewriting

2.1 Basic notions and results

In this subsection, we introduce the basics of *topological rewriting theory* as originally presented in [13]. It is a generalisation of classical rewriting theory in which we study the reduction relation with respect to a topology on the base set.

Definition 18 (Topological rewriting system) An *(abstract) topological rewriting system* is the data of $(X, \tau, \longrightarrow)$ where (X, τ) is a topological space and $\longrightarrow$ is a binary relation on X called the *base rewriting* or *one-step reduction* relation.

As for the classical setting, we study the system not only through the base relation but also via super-relations of the base relation. For instance, in the classical context, we take interest in the transitive reflexive closure of the base relation, which we denote by $\xrightarrow{\star}$. This relation describes a finite sequence of elementary computations. In topological rewriting systems, we investigate a more general kind of relation, namely the *topological rewriting relation* as defined thereafter.

Definition 19 Let $(X, \tau, \longrightarrow)$ be a topological rewriting system. The *topological rewriting relation* of that system, denoted $\relbar\!\!\oplus$, is defined as the topological closure of the relation $\xrightarrow{\star}$ viewed as a subspace of $X \times X$ equipped with the product topology $\tau_X^{\mathrm{dis}} \times \tau$, where τ_X^{dis} is the discrete topology on X.

Before noticing that this is indeed a generalisation of classical abstract rewriting systems, we will characterise in the following proposition that two elements $a, b \in X$ are related through that topological rewriting relation as $a \relbar\!\!\oplus b$ if, and only if, a rewrites in finitely many steps arbitrarily close to b (in the sense of the topology τ), *i.e.* there are finite sequences of elementary computations starting at a that come arbitrarily close to b.

Proposition 20 *Let $(X, \tau, \longrightarrow)$ be a topological rewriting system. Then, for all $a, b \in X$, we have $a \relbar\!\!\oplus b$ if, and only if, for all neighbourhoods U of b in (X, τ), there exists $c \in U$ such that $a \xrightarrow{\star} c$.*

Proof Let $\mathfrak{X}^2$ be the topological space $X \times X$ endowed with the product topology $\tau_X^{\mathrm{dis}} \times \tau$.

Suppose (a, b) is in the closure of $\xrightarrow{\star}$ when viewed as a subset of $\mathfrak{X}^2$. Let U be a neighbourhood of b in (X, τ). Then $\{a\} \times U$ is a neighbourhood of (a, b) in $\mathfrak{X}^2$ and therefore, there exists $(a, c) \in \{a\} \times U$ such that $a \xrightarrow{\star} c$.

Conversely, assume that for all neighbourhoods U of b in (X, τ) we have $c \in U$ such that it satisfies $a \xrightarrow{\star} c$. But, by definition of the product space, for any neighbourhood V of (a, b) in $\mathfrak{X}^2$, there exist a neighbourhood W of a in $\left(X, \tau_X^{\mathrm{dis}}\right)$ and a neighbourhood U of b in (X, τ) such that they satisfy the relation $W \times U \subseteq V$. Now, by assumption, there exists $c \in U$ such that $a \xrightarrow{\star} c$ which means exactly that (a, b) is in the topological closure of $\xrightarrow{\star}$ when viewed as a subset of $\mathfrak{X}^2$, since $a \in W$. $\square$

Remark 21 Note how $\xrightarrow{\star}$ is always a subrelation of $\mathrel{-\!\!\circledast}$. Now, conversely, notice that, when we consider $\tau := \tau_X^{\mathrm{dis}}$, it follows from Proposition 20 that the relation $\mathrel{-\!\!\circledast}$ associated to the topological rewriting system $\left(X, \tau_X^{\mathrm{dis}}, \longrightarrow\right)$ is exactly equal to $\xrightarrow{\star}$ as binary relations. In that sense, the topological rewriting systems are generalisations of abstract rewriting systems because most of the subsequent definitions in topological rewriting theory turn out to translate back to well-known properties of classical rewriting theory when considering the discrete topology.

The notion of normal forms remains the same as in the classical setting.

Definition 22 (Normal form) Let $(X, \tau, \longrightarrow)$ be a topological rewriting system. Any $a \in X$ is called a *normal form* of the system if $\{b \in X \mid a \longrightarrow b\} = \emptyset$.

It is possible to characterise normal forms in terms of the topological rewriting relation when we allow ourselves to assume a property on the underlying topological space. Recall that a T_1-space is a topological space (X, τ) such that for all $x \in X$, the intersection of all neighbourhoods of x in (X, τ) is exactly equal to the singleton $\{x\}$.

Lemma 23 *Let $(X, \tau, \longrightarrow)$ be a topological rewriting system.*

1. *Assume that (X, τ) is a T_1-space. Then, for any normal form a of the system and for all $b \in X$, if $a \mathrel{-\!\!\circledast} b$ then $b = a$.*
2. *Assume that $\longrightarrow$ is anti-reflexive and let $a \in X$. If, for all $b \in X$ such that $a \mathrel{-\!\!\circledast} b$ we have $b = a$, then a is a normal form for the system.*

Proof For statement number 1, assume (X, τ) is a T_1-space and let a be a normal form and consider $b \in X$ such that $a \mathrel{-\!\!\circledast} b$. Then, by Proposition 20, for all neighbourhoods U of b for the topology τ there exists $c \in U$ such that $a \xrightarrow{\star} c$. Now, since a is a normal form, $c = a$. In other words, a is contained in all neighbourhoods of b. Finally, since (X, τ) is a T_1-space by assumption, it follows that $b = a$.

For statement number 2, assume $\longrightarrow$ anti-reflexive and let $a \in X$ be an element such that $\{b \in X \mid a \mathrel{-\!\!\circledast} b\} = \{a\}$. By contradiction, assume there exists $b \in X$ such that $a \longrightarrow b$. Since $\longrightarrow$ is a subrelation of $\mathrel{-\!\!\circledast}$, we have $a \mathrel{-\!\!\circledast} b$. By assumption, it follows that $b = a$, hence $a \longrightarrow a$, which contradicts the anti-reflexivity. $\square$

One of the goals of topological rewriting theory is to study systems where classical confluence fails to be strictly verified, as well as systems which are not *terminating* (*a.k.a. strongly normalising*). However, it is still useful to consider systems in which a normal form can be reached through a super-relation of $\longrightarrow$, starting at any element of the base space. This is the purpose of the following definition.

Definition 24 Let $(X, \tau, \longrightarrow)$ be a topological rewriting system and let $\rightsquigarrow$ be a subrelation of $\mathrel{-\!\!\circledast}$. Denote by $\stackrel{\star}{\rightsquigarrow}$ the transitive reflexive closure of $\rightsquigarrow$. We say that the system is *normalising with respect to* $\rightsquigarrow$ if, for all $a \in X$ there exists a normal form b of the system such that $a \stackrel{\star}{\rightsquigarrow} b$, in which case we say "$b$ is a normal form of a reached by $\rightsquigarrow$".

Rewriting theory is especially concerned with the existence and uniqueness of normal forms of arbitrary elements of the base space as reached by a certain super-relation of the base rewriting relation. Definition 24 gives the existence and we now give three definitions related to the uniqueness of normal forms.

Definition 25 Let $(X, \tau, \longrightarrow)$ be a topological rewriting system and let $\rightsquigarrow$ be a subrelation of $\mathrel{-\!\!\circledast}$. Let $\stackrel{\star}{\rightsquigarrow}$ be the transitive reflexive closure of $\rightsquigarrow$ and $\stackrel{\star}{\leftrightsquigarrow}$ the equivalence relation generated by $\rightsquigarrow$. We say that the system:

1. has the *normal form property with respect to* $\rightsquigarrow$ if, for all $a \in X$ and any normal form b of the system such that $a \stackrel{\star}{\leftrightsquigarrow} b$, then $a \stackrel{\star}{\rightsquigarrow} b$.
2. has the *unique normal form property with respect to* $\rightsquigarrow$ if, for all normal forms a and b of the system such that $a \stackrel{\star}{\leftrightsquigarrow} b$, then $a = b$.
3. has *unique normal forms reached by* $\rightsquigarrow$ if, for all $a \in X$ and any normal forms b and c of the system such that $a \stackrel{\star}{\rightsquigarrow} b$ and $a \stackrel{\star}{\rightsquigarrow} c$, then $b = c$.

It turns out that, under a specific normalisation assumption and over a T_1-space, these three properties are equivalent. This is the content of the following proposition.

Proposition 26 *Let $(X, \tau, \longrightarrow)$ be a topological rewriting system and let $\rightsquigarrow$ be a subrelation of $\mathrel{-\!\!\circledast}$. Consider all the definitions with respect to $\rightsquigarrow$. We then have:*

1. *The unique normal form property implies that the system has unique normal forms reached by $\rightsquigarrow$.*
2. *If (X, τ) is a T_1-space and if $\mathrel{-\!\!\circledast}$ is transitive, then the normal form property implies the unique normal form property.*
3. *If the system is normalising, then the unique normal form property implies the normal form property.*
4. *If (X, τ) is a T_1-space and if the system is normalising, then, when the system has unique normal forms reached by $\rightsquigarrow$, it has the unique normal form property.*

Proof Statements 1 and 2 are straightforward with what precedes.

For statement 3, let $a \in X$ and b be a normal form of the system such that $a \stackrel{\star}{\leftrightsquigarrow} b$. By normalisation hypothesis, there exists a normal form c of the system such that $a \stackrel{\star}{\rightsquigarrow} c$. Hence, it follows that $c \overset{\star}{\leftrightsquigarrow} b$ where c and b are normal forms. Hence, by assumption, it follows that we have $b = c$ and, thus, $a \overset{\star}{\rightsquigarrow} b$, the desired result.

For statement 4, let a and b be normal forms of the system such that $a \overset{\star}{\leftrightsquigarrow} b$. Denote by $\leftsquigarrow$ and $\overset{\star}{\leftsquigarrow}$ the opposite relations of $\rightsquigarrow$ and $\overset{\star}{\rightsquigarrow}$ respectively. If we let $\leftrightsquigarrow$ be the symmetric closure of $\rightsquigarrow$, the assertion $a \overset{\star}{\leftrightsquigarrow} b$ exactly means that there exists a finite tuple of elements $(c_1, \cdots, c_\ell) \in X^\ell$, for $\ell \in \mathbb{N}$, such that:

$$a \leftrightsquigarrow c_1 \leftrightsquigarrow c_2 \leftrightsquigarrow \cdots \leftrightsquigarrow c_\ell \leftrightsquigarrow b.$$

Let us call a "valley" in a sequence $c_1 \leftrightsquigarrow c_2 \leftrightsquigarrow \cdots \leftrightsquigarrow c_\ell$ any index $i \in \{2, \cdots, \ell - 1\}$ such that $c_{i-1} \rightsquigarrow c_i \leftsquigarrow c_{i+1}$.

Let us show that, for any $\ell \in \mathbb{N}$ and any tuple $(c_1, \cdots, c_\ell) \in X^\ell$ such that:

$$a \leftsquigarrow c_1 \leftrightsquigarrow c_2 \leftrightsquigarrow \cdots \leftrightsquigarrow c_\ell \rightsquigarrow b.$$

which contains $\nu \in \mathbb{N} \setminus \{0\}$ valleys, there exists $\ell' \in \mathbb{N}$ and $(d_1, \cdots, d_{\ell'}) \in X^{\ell'}$ such that:

$$a \leftsquigarrow d_1 \leftrightsquigarrow d_2 \leftrightsquigarrow \cdots \leftrightsquigarrow d_{\ell'} \rightsquigarrow b.$$

which contains $\nu - 1$ valleys.

Indeed, if there are $\nu \geq 1$ valleys in the first sequence, there is a right-most one, by which we mean, the biggest index $i \in \{2, \cdots, \ell - 1\}$ which is a valley. In other words, we have the following diagram:

$$c_{i-1} \rightsquigarrow c_i \leftsquigarrow c_{i+1} \leftsquigarrow c_{i+2} \leftsquigarrow \cdots \leftsquigarrow c_j \rightsquigarrow c_{j+1} \rightsquigarrow \cdots \rightsquigarrow c_\ell \rightsquigarrow b,$$

for a unique $j \in \{i + 2, \cdots, \ell\}$.

By normalisation hypothesis, there exists a normal form d for the system that satisfies the relation $c_i \overset{\star}{\rightsquigarrow} d$. Note how we are thus in the following situation:

$$c_{i-1} \overset{\star}{\rightsquigarrow} d \overset{\star}{\leftsquigarrow} c_j \overset{\star}{\rightsquigarrow} b.$$

By assumption of unique normal forms reached by $\rightsquigarrow$, it follows that $b = d$. Thus, there exist $\ell'' \in \mathbb{N}$ and $(d_{i+k})_{0 \leq k \leq \ell''}$ such that $c_{i-1} \rightsquigarrow d_i \rightsquigarrow d_{i+1} \rightsquigarrow \cdots \rightsquigarrow d_{i+\ell''} \rightsquigarrow d = b$.

It then suffices to define $\ell' := i + \ell''$ and $d_k := c_k$ for all $k \in \{1, \cdots, i - 1\}$ and it follows that the sequence:

$$a \leftsquigarrow d_1 \leftrightsquigarrow d_2 \leftrightsquigarrow \cdots \leftrightsquigarrow d_{i-1} \rightsquigarrow d_i \rightsquigarrow d_{i+1} \rightsquigarrow \cdots \rightsquigarrow d_{i+\ell''} = d_{\ell'} \rightsquigarrow b.$$

contains $\nu - 1$ valleys.

Now, since $\rightsquigarrow$ is a subrelation of $\mathrel{-\!\!\!\bigcirc\kern-0.9em\to}$, for all $c \in X$ such that $a \leftrightsquigarrow c$, we have:

$$a \mathrel{-\!\!\!\bigcirc\kern-0.9em\to} c \quad \text{if} \quad c \rightsquigarrow a \,\text{is false.} \tag{2}$$

However, by hypothesis, the space (X, τ) is a T_1-space, and thus, by Lemma 23 and since a is a normal form, it follows that (2) yields the following:

$$c = a \quad \text{if} \quad c \rightsquigarrow a \,\text{is false.}$$

Similarly, for any $c \in X$ such that $c \leftrightsquigarrow b$, we have $c = b$ if $c \rightsquigarrow b$ is false. Utilising the contrapositive, this means that, for any sequence and $\ell \geq 1$:

$$a \leftrightsquigarrow c_1 \leftrightsquigarrow c_2 \leftrightsquigarrow \cdots \leftrightsquigarrow c_\ell \leftrightsquigarrow b.$$

there exist $i_0, i_1 \in \{1, \cdots, \ell\}$ such that $i_0 \leq i_1$:

$$a \leftsquigarrow c_{i_0} \leftrightsquigarrow c_{i_0+1} \leftrightsquigarrow c_{i_0+2} \leftrightsquigarrow \cdots \leftrightsquigarrow c_{i_1-1} \leftrightsquigarrow c_{i_1} \rightsquigarrow b$$

Therefore, since $a \overset{\star}{\leftrightsquigarrow} b$, there necessarily exist $\ell \in \mathbb{N}$ and a tuple $(c_1, \cdots, c_\ell) \in X^\ell$ such that:

$$a \leftsquigarrow c_1 \leftrightsquigarrow c_2 \leftrightsquigarrow \cdots \leftrightsquigarrow c_\ell \rightsquigarrow b.$$

which, moreover, contains no valley, thanks to the process given earlier. This means that there exists $i \in \{1, \cdots, \ell\}$ verifying:

$$a \leftsquigarrow c_1 \leftsquigarrow c_2 \leftsquigarrow \cdots \leftsquigarrow c_i \rightsquigarrow c_{i+1} \rightsquigarrow \cdots \rightsquigarrow c_\ell \rightsquigarrow b,$$

At this point, it suffices to use the assumption of Definition 25-3 to conclude that $a = b$ since the above sequence implies $c_i \overset{\star}{\rightsquigarrow} a$ and $c_i \overset{\star}{\rightsquigarrow} b$. □

We now introduce two different notions of confluence for topological rewriting systems, which are exactly the classical confluence when considering the discrete topology.

Definition 27 (Topological confluence) Let $(X, \tau, \longrightarrow)$ be a topological rewriting system. We say that the system is:

1. *finitary topologically confluent* if, for all $a, b, c \in X$ such that $a \overset{\star}{\longrightarrow} b$ and $a \overset{\star}{\longrightarrow} c$, then there exists $d \in X$ such that $b \multimap d$ and $c \multimap d$. In diagrams:

$$\begin{tikzcd} & a \arrow[dl, "\star"'] \arrow[dr, "\star"] & \\ b \arrow[dr, dashed, multimap] & & c \arrow[dl, dashed, multimap] \\ & d & \end{tikzcd}$$

2. *infinitary topologically confluent* if, for all $a, b, c \in X$ such that $a \multimap b$ and $a \multimap c$, then there exists $d \in X$ such that $b \multimap d$ and $c \multimap d$. In diagrams:

$$\begin{tikzcd} & a \arrow[dl, multimap] \arrow[dr, multimap] & \\ b \arrow[dr, dashed, multimap] & & c \arrow[dl, dashed, multimap] \\ & d & \end{tikzcd}$$

Remark 28 Since $\overset{\star}{\longrightarrow}$ is a subrelation of $\multimap$, it is clear that infinitary topological confluence implies finitary topological confluence of the system. See [17, Examples 4.1.3, 4.1.4] for counter-examples of the converse implication.

2.2 Attractivity of normal forms in metric spaces

We described in Subsection 2.1 the basic definitions of topological rewriting theory, including the notion of *normal forms* of a system. As we will see in Subsection 3.1, our primary example of a non-discrete topological rewriting system, based on formal power series, enjoys the property that the distance between any arbitrary starting formal power series f and any normal form of the system will always be greater or equal to the distance between any of the successors of f and the considered normal form. In other words, rewriting cannot take us further away from any normal forms than we already were. Intuitively, this property seems to describe a certain kind of attractivity of normal forms; we formalise that concept in terms of topological rewriting systems for which the underlying space is a metric space.

Definition 29 Let $(X, \tau_{\mathrm{d}}, \longrightarrow)$ be a topological rewriting system where (X, τ_{d}) is the topological space induced by a metric space (X, d). We say that the system has *globally attractive normal forms* if, for any normal form c of the system and any $a, b \in X$ such that $a \multimap b$, we have $\mathrm{d}(b, c) \leq \mathrm{d}(a, c)$.

Remark 30 Note how this is equivalent to saying:

$$\text{if } a \longrightarrow b, \text{ then } \mathrm{d}(b, c) \leq \mathrm{d}(a, c), \text{ for any } a, b \in X \text{ and any normal form } c. \tag{3}$$

Indeed, the necessity is immediate since $\longrightarrow$ is a subrelation of $\multimap$. Now, assume (3). Consider $a, b \in X$ such that $a \multimap b$ and let c be a normal form of the system. First, we show that for any $\varepsilon \in \mathbb{R}_{>0}$, there exists $d \in X$ such that $\mathrm{d}(d, b) \leq \varepsilon$ and $\mathrm{d}(d, c) \leq d(a, c)$: since we have $a \multimap b$ and the open ball $\mathrm{B}_\varepsilon(b)$ centered on b of radius ε is a neighbourhood of b, there exists $d_\varepsilon \in \mathrm{B}_\varepsilon(b)$ such that $a \xrightarrow{\star} d_\varepsilon$. Decomposing that transitive reflexive relation into a sequence of one-step reduction relations, we obtain, by induction and by (3), the following inequality $\mathrm{d}(d_\varepsilon, c) \leq \mathrm{d}(a, c)$. Hence, for any $\varepsilon \in \mathbb{R}_{>0}$ we have:

$$\mathrm{d}(b, c) \leq \mathrm{d}(b, d_\varepsilon) + \mathrm{d}(d_\varepsilon, c) \leq \varepsilon + \mathrm{d}(a, c).$$

Making ε tend towards 0 we obtain the desired result: $\mathrm{d}(b, c) \leq \mathrm{d}(a, c)$.

In the next subsection, we will use this newly-defined property to prove that, under certain assumptions, finitary topological confluence implies the normal form property with respect to any subrelation of $\multimap$.

2.3 When finitary topological confluence implies uniqueness of normal forms

Throughout this subsection, fix $(X, \tau_{\mathrm{d}}, \longrightarrow)$ a topological rewriting system where the underlying topological space (X, τ_{d}) is induced by a metric space (X, d) and $\rightsquigarrow$ a subrelation of $\multimap$. Denote as usual $\overset{\star}{\rightsquigarrow}$ the transitive reflexive closure of $\rightsquigarrow$.

The main theorem of the current subsection is stated as follows:

Theorem 31 *Assume the following properties:*

1. *The relation $\multimap$ is transitive.*
2. *The system is normalising with respect to $\rightsquigarrow$.*
3. *The system has globally attractive normal forms.*

Then, if the system is finitary topologically confluent, then it has the unique normal form property with respect to $\rightsquigarrow$.

Proof Since (X, τ_{d}) is a metrisable space, it is Hausdorff and hence a T_1-space as well. With that in mind and using Assumption 2, by Proposition 26-4, it suffices to show that the system has unique normal forms reached by $\rightsquigarrow$ (Definition 25-3) to prove our theorem. Thus, consider $x \in X$ and α, α' be two normal forms such that $x \overset{\star}{\rightsquigarrow} \alpha$ and $x \overset{\star}{\rightsquigarrow} \alpha'$. Our objective is $\alpha = \alpha'$, let us prove it by showing that $\mathrm{d}(\alpha, \alpha')$ is bounded from above by a quantity that converges to 0. Since $\rightsquigarrow$ is a subrelation of $\xrightarrow{\circ}$ and by Assumption 1 that latter relation is transitive, the relation $\overset{\star}{\rightsquigarrow}$ is a subrelation of $\xrightarrow{\circ}$ as well. Hence, $x \xrightarrow{\circ} \alpha$ and $x \xrightarrow{\circ} \alpha'$. Now, by Proposition 20, since the open balls $\mathrm{B}_\rho(\alpha)$ and $\mathrm{B}_\rho(\alpha')$ of radius $\rho \in \mathbb{R}_{>0}$ and centered at α and α' are neighbourhoods of α and α' respectively, there exists $(y, y') \in \mathrm{B}_\rho(\alpha) \times \mathrm{B}_\rho(\alpha')$ such that $x \overset{\star}{\rightarrow} y$ and $x \overset{\star}{\rightarrow} y'$. By the finitary topological confluence hypothesis, it follows that there exists $z \in X$ such that $y \xrightarrow{\circ} z$ and $y' \xrightarrow{\circ} z$. Now, by Assumption 3, we have the inequalities $\mathrm{d}(z, \alpha) \leq \mathrm{d}(y, \alpha)$ and $\mathrm{d}(z, \alpha') \leq \mathrm{d}(y', \alpha')$. Therefore, we have:

$$\mathrm{d}(\alpha, \alpha') \leq \mathrm{d}(\alpha, z) + \mathrm{d}(z, \alpha') \leq \mathrm{d}(y, \alpha) + \mathrm{d}(y', \alpha') \leq 2\rho.$$

Hence, making ρ tend towards 0 we obtain $\alpha = \alpha'$, the desired result. □

For our purposes of Subsection 3.3, we will require the following corollary:

Corollary 32 *Under the same Assumptions 1,2,3 of Theorem 31, if the system is finitary topologically confluent, then it has the normal form property with respect to the relation* $\rightsquigarrow$.

Proof This is a direct consequence of Theorem 31 and Statement 3 of Proposition 26. □

3 Characterisations of standard bases in formal power series ideals

In this section, we will first recall in Subsection 3.1 how rewriting with respect to an ideal of formal power series works. This is essentially the same process as multivariate polynomial reduction but with respect to the opposite order of an admissible monomial order. We will also prove some rewriting-theoretic properties of the induced system. Then, in Subsection 3.2, we will introduce the language which has originally been used to characterise standard bases [24], namely *standard representations* and *Hironaka remainder*. Finally, in Subsection 3.3, we will extend those previous characterisations by adding some equivalent statements in terms of topological rewriting theory. In particular, we will recover the result of [17] stating that finitary and infinitary topological confluences are equivalent for formal power series as well as the result of [13] characterising standard bases with finitary topological confluence (called δ-confluence in that paper). Moreover, it will be made clear that the congruence relation modulo the ideal is nothing else than the equivalence relation generated by the topological rewriting relation induced by the topological rewriting system from Subsection 3.1.

Throughout Subsections 3.1-3.2-3.3, $\mathbb{K}$ is a (commutative) field of arbitrary characteristic, n is a positive integer,$x_1, \cdots, x_n$ are distinct indeterminates and, as defined in Subsection 1.2, denote by $[x_1, \cdots, x_n]$ (or $[\mathbf{x}]$ for short) the commutative free monoid generated by $\{x_1, \cdots, x_n\}$ and by $\mathbb{K}[[x_1, \cdots, x_n]]$ (or $\mathbb{K}[[\mathbf{x}]]$ for short) the $\mathbb{K}$-algebra of formal power series. Consider also a fixed admissible monomial order $<$ on $[\mathbf{x}]$. Recall from Subsection 1.2, that $\mathbb{K}[[\mathbf{x}]]$ is a complete local equicharacteristic Noetherian ring whose $I(\mathbf{x})$-adic topology is metrisable by the metric δ defined in (1). In particular, the pair $(\mathbb{K}[[\mathbf{x}]], \delta)$ is a complete metric space. Let τ_δ be the induced topology by the metric δ. Finally, fix a set $R := \{s_1, \cdots, s_r\} \subseteq \mathbb{K}[[\mathbf{x}]] \setminus \{0\}$ of exactly r non-zero formal power series, where r is a non-negative integer and denote by $I(R)$ the ideal in $\mathbb{K}[[\mathbf{x}]]$ generated by R.

3.1 Rewriting on formal power series

Define the following binary relation: for $f, g \in \mathbb{K}[[\mathbf{x}]]$, we write $f \longrightarrow g$ if there exist a monomial $M \in \operatorname{supp}(f)$, an index $i \in \{1, \cdots, r\}$ and $m \in [\mathbf{x}]$ such that $M = m \cdot \operatorname{lm}(s_i)$ and:

$$g = f - \frac{\langle f \mid M \rangle}{\operatorname{lc}(s_i)} m s_i.$$

Notice how that means that $g = (f - \langle f \mid M \rangle M) + (\text{terms} > M)$. In other words, f and g have the same coefficients for any monomial that is strictly smaller than M, and the coefficient for M in g is necessarily zero.

The relation $\longrightarrow$ thus depends on R and $<$. We can now consider the topological rewriting system $\mathfrak{X}_R^< := (\mathbb{K}[[\mathbf{x}]], \tau_\delta, \longrightarrow)$ and, therefore, the topological rewriting relation (Definition 19) of that system that we will denote $\multimap$. It is clear that normal forms (Definition 22) of the system $\mathfrak{X}_R^<$ are exactly the formal power series f such that no monomial in $\operatorname{supp}(f)$ is a multiple of a monomial of the form $\operatorname{lm}(s_i)$ for any index $i \in \{1, \cdots, r\}$. Denote by $\operatorname{NF}(\mathfrak{X}_R^<)$ the set of normal forms of the system. Let us characterise the topological rewriting relation in terms of converging sequences.

Lemma 33 *For any $f, g \in \mathbb{K}[[\mathbf{x}]]$, we have:*

$$f \multimap g \qquad \Longleftrightarrow \qquad \exists (h_k)_{k \in \mathbb{N}} \in \mathbb{K}[[\mathbf{x}]]^{\mathbb{N}}, \begin{cases} f \xrightarrow{\star} h_k & \forall k \in \mathbb{N}, \\ \lim_{k \to \infty} h_k = g. \end{cases}$$

Proof The right-to-left direction is trivial according to Proposition 20. Now, consider the countable fundamental system of neighbourhoods of g defined as $(\mathrm{B}_{2^{-k}}(g))_{k \in \mathbb{N}}$, where the notation $\mathrm{B}_{2^{-k}}(g)$ is the open ball for the metric δ of radius 2^{-k} centered at g. Then, if $f \multimap g$, there exists, for each $k \in \mathbb{N}$, at least one $h_k \in \mathrm{B}_{2^{-k}}(g)$ such that $f \xrightarrow{\star} h_k$. Now, by the axiom of choice, it suffices to make arbitrary choices of such h_k for each $k \in \mathbb{N}$ to construct a sequence that satisfies the right-hand side of the equivalence of the lemma. □

Remark 34 Note that this characterisation can be applied *mutatis mutandis* to any topological rewriting system whose underlying topological space is *first-countable*, that is to say, every element of the space has a countable fundamental system of neighbourhoods.

Here is one important property of the system $\mathfrak{X}_R^<$:

Proposition 35 *If the order $<$ is compatible with the degree, then $\multimap$ is transitive.*

Proof Let $f, g, h \in \mathbb{K}[[\mathbf{x}]]$. We will proceed in three steps.

- First, we will show that if $f \mathrel{-\!\!\circledast} g \longrightarrow h$, then $f \mathrel{-\!\!\circledast} h$.
- Second, it will follow that if $f \mathrel{-\!\!\circledast} g \xrightarrow{\star} h$, then $f \mathrel{-\!\!\circledast} h$.
- Finally, we will use that intermediate result to prove that, if $f \mathrel{-\!\!\circledast} g \mathrel{-\!\!\circledast} h$, we indeed have $f \mathrel{-\!\!\circledast} h$, hence $\mathrel{-\!\!\circledast}$ is transitive.

Thus, assume firstly that $f \mathrel{-\!\!\circledast} g \longrightarrow h$. By definition of $g \longrightarrow h$, there exist $M \in \operatorname{supp}(g)$, an index $i \in \{1, \cdots, r\}$ and a monomial $m \in [\mathbf{x}]$ such that $M = m \cdot \operatorname{lm}(s_i)$ and:

$$h = g - \frac{\langle g \,|\, M \rangle}{\operatorname{lc}(s_i)} m s_i.$$

From that equality, it follows that $\operatorname{lm}(g - h) = M$. Let us use Proposition 20 to show that $f \mathrel{-\!\!\circledast} h$. Let U be an open neighbourhood of h. If g is in U, then there exists $h' \in U$ such that $f \xrightarrow{\star} h'$ since $f \mathrel{-\!\!\circledast} g$, Otherwise, assume that $g \notin U$. Therefore, there exists $N_U \in \mathbb{N}$ such that $\mathrm{B}_{2^{-N_U}}(h)$ is an open ball contained in U and, thus, does not contain g. It follows from Proposition 15, since the order is, by hypothesis, compatible with the degree, that $\deg(\operatorname{lm}(g - h)) = \deg(M) \leq N_U$. By Lemma 33, there exists a sequence $(h_k)_{k \in \mathbb{N}}$ such that $f \xrightarrow{\star} h_k$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} h_k = g$. It follows that there exists $K_{N_U} \in \mathbb{N}$ verifying $\delta\left(h_{K_{N_U}}, g\right) < 2^{-N_U}$. Since the order is compatible with the degree, we obtain by Proposition 15 again:

$$\deg\left(\operatorname{lm}\left(g - h_{K_{N_U}}\right)\right) > N_U \geq \deg(M) = \deg(\operatorname{lm}(g - h)).$$

By compatibility with the degree, we obtain $\operatorname{lm}\left(g - h_{K_{N_U}}\right) > \operatorname{lm}(g - h)$ and, therefore, $\left\langle g - h_{K_{N_U}} \,\middle|\, M \right\rangle = 0$, thus, $\left\langle h_{K_{N_U}} \,\middle|\, M \right\rangle = \langle g \,|\, M \rangle \neq 0$. We can therefore define:

$$h' := h_{K_{N_U}} - \frac{\left\langle h_{K_{N_U}} \,\middle|\, M \right\rangle}{\operatorname{lc}(s_i)} m s_i$$

which by construction verifies $h_{K_{N_U}} \longrightarrow h'$. Now we compute:

$$\begin{aligned} h - h' &= h - h_{K_{N_U}} + \frac{\left\langle h_{K_{N_U}} \,\middle|\, M \right\rangle}{\operatorname{lc}(s_i)} m s_i \\ &= \left(h + \frac{\langle g \,|\, M \rangle}{\operatorname{lc}(s_i)} m s_i\right) - h_{K_{N_U}} \\ &= g - h_{K_{N_U}}. \end{aligned}$$

Thus, trivially $\delta(h, h') = \delta\left(g, h_{K_{N_U}}\right) < 2^{-N_U}$. Hence, $h' \in \mathrm{B}_{2^{-N_U}}(h) \subseteq U$. Moreover, by construction, we have $f \xrightarrow{\star} h_{K_{N_U}} \longrightarrow h'$. Therefore, we did exhibit $h' \in U$, for U arbitrary open neighbourhood of h, such that $f \xrightarrow{\star} h'$. Thus, $f \mathrel{-\!\!\circledast} h$.

Secondly, by a simple induction argument, if $f \mathrel{-\!\!\circledast} g \xrightarrow{\star} h$, we have $f \mathrel{-\!\!\circledast} h$ since we decompose $g \xrightarrow{\star} h$ into a finite sequence of one-step reduction relations and thus can apply step-by-step the first result of the proof.

Thirdly, assume $f \mathrel{-\!\!\circledast} g \mathrel{-\!\!\circledast} h$. By Lemma 33, there exists a sequence $(h_k)_{k \in \mathbb{N}}$ such that, for any $k \in \mathbb{N}$, $g \xrightarrow{\star} h_k$ and $\lim_{k \to \infty} h_k = h$. Let us show that $f \mathrel{-\!\!\circledast} h$ by proving that, for any $\varepsilon \in \mathbb{R}_{>0}$, there exists $h' \in \mathrm{B}_\varepsilon(h)$ such that $f \xrightarrow{\star} h'$ which enables to conclude via Proposition 20. Let $\varepsilon \in \mathbb{R}_{>0}$. Then, since $(h_k)_{k \in \mathbb{N}}$ converges to h, there exists $K_\varepsilon \in \mathbb{N}$ such that $\delta\left(h, h_{K_\varepsilon}\right) < \frac{\varepsilon}{2}$. Moreover, we have $f \rightarrow\!\!\circledast\, g \xrightarrow{\star} h_{K_\varepsilon}$. Hence, by the second result of this proof, we have $f \rightarrow\!\!\circledast\, h_{K_\varepsilon}$. Using Lemma 33 again, there exists a sequence $(h'_k)_{k\in\mathbb{N}}$ such that, for all $k \in \mathbb{N}$, $f \xrightarrow{\star} h'_k$ and $\lim_{k\to\infty} h'_k = h_{K_\varepsilon}$. By that latter convergence property, there exists $K'_\varepsilon \in \mathbb{N}$ such that $\delta\left(h_{K_\varepsilon}, h'_{K'_\varepsilon}\right) < \frac{\varepsilon}{2}$. Hence:

$$\delta\left(h, h'_{K'_\varepsilon}\right) \leq \delta\left(h, h_{K_\varepsilon}\right) + \delta\left(h_{K_\varepsilon}, h'_{K'_\varepsilon}\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, by definition of the sequence $(h'_k)_{k\in\mathbb{N}}$, we have $f \xrightarrow{\star} h'_{K'_\varepsilon}$. Which concludes the proof that $f \rightarrow\!\!\circledast\, h$.

□

We will now prove a useful lemma.

Lemma 36 *Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{K}[[\mathbf{x}]]$. Assume the monomial order $<$ is compatible with the degree. If, for all $k \in \mathbb{N}$, we have $f_k \longrightarrow f_{k+1}$, then the sequence $(f_k)_{k\in\mathbb{N}}$ is convergent in $\mathbb{K}[[\mathbf{x}]]$.*

Proof For each $k \in \mathbb{N}$, take the monomial $M_k \in \operatorname{supp}(f_k)$, the index $i_k \in \{1, \cdots, r\}$ and the monomial $m_k \in [\mathbf{x}]$ such that $M_k = m_k \cdot \operatorname{lm}(s_{i_k})$ and $f_k - f_{k+1} = \frac{\langle f_k \mid M_k\rangle}{\operatorname{lc}(s_{i_k})} m_k s_{i_k}$.

Firstly, let us show that, for any $k_1 \in \mathbb{N}$, there exists $k' > k_1$ such that $M_{k_1} < M_k$ for all $k \geq k'$. Let $k_1 \in \mathbb{N}$. Let $M := M_{k_1}$ and define $L_1 := \{M_k \mid k > k_1, M_k \leq M\}$. Since there are only finitely many indeterminates and the order is compatible with the degree, L_1 is finite because M is an upper bound. If L_1 is empty, then all the subsequent M_k for $k > k_1$ are then bigger than M which is the desired result. Assume thus L_1 non-empty and denote by $M_1^{\min} := \min L_1$ and $k_2 := \min\left\{k \in \mathbb{N} \,\middle|\, k > k_1, M_k = M_1^{\min}\right\}$. By definition, $M_1^{\min}$ is the monomial in $\operatorname{supp}\left(f_{k_2}\right)$ that is reduced at step k_2. Hence, it follows that $\left\langle f_k \,\middle|\, M_1^{\min}\right\rangle = 0$ for all $k > k_2$ since $M_1^{\min}$ is the minimum of the monomials reduced after step k_1. Then, consider the subset $L_2 := \{M_k \mid k > k_2, M_k \leq M\} \subseteq L_1$ and notice how $M_1^{\min} \notin L_2$ by the previous discussion. Repeat this process *ad infinitum* replacing the index j by $j+1$ in the definitions of L_{j+1}, $M_{j+1}^{\min}$ and k_{j+2}. Notice that, at each step j, we have $L_{j+1} \subsetneq L_j$ because $M_j^{\min} \notin L_{j+1}$. But since L_1 is finite, there necessarily exists $j \geq 1$ such that we have $L_{j+1} := \left\{M_k \,\middle|\, k > k_{j+1}, M_k \leq M_{k_1}\right\} = \emptyset$. Finally, this implies that $M_{k_{j+1}+\ell} \notin L_1$, and thus, $M_{k_{j+1}+\ell} > M_{k_1}$, for all $\ell \geq 1$.

Now, let $D \in \mathbb{N}$. Since the order is compatible with the degree and using the previous part of this proof, the sequence $(\deg(M_k))_{k\in\mathbb{N}}$ is eventually bigger than D, *i.e.*, there exists a rank $k_D \in \mathbb{N}$ such that $\deg(M_k) > D$ for all $k \geq k_D$. But, a straightforward computation shows that, for $k_1, k_2 \in \mathbb{N}$ with $k_D \leq k_1 < k_2$:

$$\deg\left(\operatorname{lm}\left(f_{k_1} - f_{k_2}\right)\right) = \min\left\{\deg(M_k) \mid k_1 \leq k < k_2\right\} > D.$$

By compatibility with the degree, we use Proposition 15 to show that $\delta(f_{k_1}, f_{k_2}) < 2^{-D}$ which in turn proves that $(f_k)_{k\in\mathbb{N}}$ is Cauchy. Now, since $\mathbb{K}[[\mathbf{x}]]$ is complete as metric space, it follows that the sequence $(f_k)_{k\in\mathbb{N}}$ does indeed converge to a limit in $\mathbb{K}[[\mathbf{x}]]$. □

Remark 37 Denote by $\xrightarrow{=}$ the reflexive closure of $\longrightarrow$, *i.e.*, $f \xrightarrow{=} g$ if and only if $f = g$ or $f \longrightarrow g$. As sequences that are eventually stationary are Cauchy sequences in a trivial way, any sequence $(f_k)_{k\in\mathbb{N}}$ such that $f_k \xrightarrow{=} f_{k+1}$ (and thus, $f_k \xrightarrow{\star} f_{k+1}$) for all $k \in \mathbb{N}$ is convergent in $\mathbb{K}[[\mathbf{x}]]$, by Lemma 36.

Now, consider the following relation that we will call the *topological rewriting with chains relation*:

$$f \rightarrowtail\!\!\!\!\!\oplus\, g \qquad \text{when} \qquad \exists (h_k)_{k\in\mathbb{N}} \in \mathbb{K}[[\mathbf{x}]]^{\mathbb{N}}, \begin{cases} h_0 = f \\ h_k \xrightarrow{=} h_{k+1} & \forall k \in \mathbb{N}, \\ \lim_{k\to\infty} h_k = g. \end{cases}$$

By Lemma 33, it is immediate that $\rightarrowtail\!\!\!\!\oplus$ is a subrelation of $\multimap\!\!\!\!\oplus$, because, if two elements are linked by a finite sequence of $\xrightarrow{=}$, then they are related through $\xrightarrow{\star}$. However, it is still an open question if $\rightarrowtail\!\!\!\!\oplus$ is actually equal to $\multimap\!\!\!\!\oplus$ in the system $\mathfrak{X}_R^<$: this is the content of the problem we call the *chains conjecture*.

Proposition 38 *If the order $<$ is compatible with the degree, then, for all $f \in \mathbb{K}[[\mathbf{x}]]$, there exists a normal form $\alpha \in \mathrm{NF}\left(\mathfrak{X}_R^<\right)$ such that $f \rightarrowtail\!\!\!\!\oplus\, \alpha$. In particular, the system $\mathfrak{X}_R^<$ is normalising with respect to $\rightarrowtail\!\!\!\!\oplus$.*

Proof For any $f \in \mathbb{K}[[\mathbf{x}]]$, denote by $R_f := \{m \cdot \mathrm{lm}(s_i) \in \mathrm{supp}(f) \mid m \in [\mathbf{x}], i \in \{1, \cdots, r\}\}$. This is the set of reducible monomials in the support of f. Obviously, we have $f \in \mathrm{NF}\left(\mathfrak{X}_R^<\right)$ if and only if $R_f = \emptyset$. Let $f \in \mathbb{K}[[\mathbf{x}]]$. Let us construct a sequence $(h_k)_{k\in\mathbb{N}}$ inductively as follows. Start by setting $h_0 := f$. Now, let $K \in \mathbb{N}$ and assume we constructed a sequence $h_0 \xrightarrow{=} h_1 \xrightarrow{=} \cdots \xrightarrow{=} h_K$ such that, for all $k \in \{0, \cdots, K-1\}$, if neither h_k nor h_{k+1} are normal forms, *i.e.* if we have $R_{h_k} \neq \emptyset \neq R_{h_{k+1}}$, then $\min_< R_{h_k} < \min_< R_{h_{k+1}}$. If $R_{h_K} = \emptyset$, define all the subsequent terms of the sequence as $h_{K+\ell} := h_K$ for all $\ell \geq 1$. Now, assume that $R_{h_K} \neq \emptyset$. Let $M_K := \min_< R_{h_K}$. It exists since the order is admissible. Choose $i_K := \min\{i \in \{1, \cdots, r\} \mid \exists m \in [\mathbf{x}], M_K = m \cdot \mathrm{lm}(s_i)\}$ and $m_K := \frac{M_K}{\mathrm{lm}(s_{i_K})}$. Define:

$$h_{K+1} := h_K - \frac{\langle h_K \mid M_K \rangle}{\mathrm{lc}(s_{i_K})} m_K s_{i_K}.$$

Notice how $h_K \longrightarrow h_{K+1}$ and, if $R_{h_{K+1}} \neq \emptyset$, then $\min_< R_{h_K} < \min_< R_{h_{K+1}}$ because we rewrote the smallest reducing monomial in $\mathrm{supp}(h_K)$ and thus, $\langle h_{K+1} \mid M_K \rangle = 0$ and every new potential reducible monomial introduced by the rewriting process is necessarily bigger than M_K.

In all cases, repeating this process *ad infinitum* yields a sequence $(h_k)_{k\in\mathbb{N}}$ that satisfies the relations $h_0 = f$ and $h_k \xrightarrow{=} h_{k+1}$ for all $k \in \mathbb{N}$. Now, by compatibility with the degree we can use Lemma 36 and Remark 37. Hence, denote the limit of $(h_k)_{k\in\mathbb{N}}$ by $g := \lim_{k\to\infty} h_k$. By definition, we have $f \rightarrowtail\!\!\!\!\oplus\, g$. Finally, notice how g is a normal form: indeed, if $R_g \neq \emptyset$, say there exists $M := m \cdot \mathrm{lm}(s_i) \in \mathrm{supp}(g)$ for some $m \in [\mathbf{x}]$ and $i \in \{1, \cdots, r\}$ and let $D := \deg(M)$. Since g is the limit of the sequence, there exists $K \in \mathbb{N}$ such that we have $\delta(h_k, g) < 2^{-D}$ for all $k \geq K$. By compatibility with the degree and Proposition 15, we obtain, for all $k \geq K$, $\mathrm{lm}(h_k - g) > M$ and hence $\langle h_k \mid M \rangle = \langle g \mid M \rangle \neq 0$, so $M \in R_{h_k}$. But, since the monomial order $<$ is admissible and compatible with the degree and the fact that we have only finitely many indeterminates, for any monomial $m \in [\mathbf{x}]$, there are only finitely many monomials that are smaller than m for $<$. However, by construction, the sequence $\left(\min_< R_{h_k}\right)_{k\in\mathbb{N}}$ is strictly increasing for $<$. Now, since we saw that our assumption entails that, for all $k \geq K$, we have $M \in R_{h_k}$ and, thus, $\min_< R_{h_k} < M$, this would mean that there are infinitely many monomial smaller than M for $<$, because the sequence $\left(\min_> R_{h_k}\right)_{k\in\mathbb{N}}$ is strictly increasing but bounded from above by M. This is a contradiction, hence g is a normal form. □

To conclude this section, we will prove, as we previously said, that the system $\mathfrak{X}_R^<$ has globally attractive normal forms (Definition 29) when the order is compatible with the degree.

Proposition 39 *If the order $<$ is compatible with the degree, the system $\mathfrak{X}_R^<$ has globally attractive normal forms.*

Proof Let $f, g \in \mathbb{K}[[\mathbf{x}]]$ and $\alpha \in \mathrm{NF}\left(\mathfrak{X}_R^<\right)$. By Remark 30, it suffices to show that, if $f \longrightarrow g$, then $\delta(g, \alpha) \leq \delta(f, \alpha)$. But, by definition, if $f \longrightarrow g$, there exists $M \in \mathrm{supp}\,(f)$, $i \in \{1, \cdots, r\}$ and $m \in [\mathbf{x}]$ such that $M = m \cdot \mathrm{lm}\,(s_i)$ and:

$$g = f - \frac{\langle f \,|\, M\rangle}{\mathrm{lc}\,(s_i)} m s_i.$$

If $g = \alpha$, the result is trivial since $\delta(g, \alpha) = 0$. Assume thus $g \neq \alpha$. Then we first want to show that $\mathrm{lm}\,(g - \alpha) \geq \mathrm{lm}\,(f - \alpha)$. Indeed, let $m' \in [\mathbf{x}]$ with $m' < \mathrm{lm}\,(f - \alpha)$, then, by definition of leading monomial, $\left\langle f \,\middle|\, m'\right\rangle = \left\langle \alpha \,\middle|\, m'\right\rangle$. Hence, if $m' \in \mathrm{supp}\,(f)$, it is irreducible. By contrapositive, we deduce that $\mathrm{lm}\,(f - \alpha) \leq M$, since M is reducible and in $\mathrm{supp}\,(f)$ by definition. But, since M is the reduced monomial in the rewrite step $f \longrightarrow g$, it follows that, for all $m' < M$, we have $\left\langle g \,\middle|\, m'\right\rangle = \left\langle f \,\middle|\, m'\right\rangle$. If we impose furthermore that $m' < \mathrm{lm}\,(f - \alpha)$, we get $\left\langle g - \alpha \,\middle|\, m'\right\rangle = \left\langle f - \alpha \,\middle|\, m'\right\rangle = 0$, which is what we wanted to first prove. Now, since the order is assumed to be compatible with the degree, we can use Proposition 15 to conclude that $\delta(g, \alpha) \leq \delta(f, \alpha)$. □

3.2 Standard representations

In this subsection, we present the language of standard representations in which standard bases were originally characterised.

Definition 40 Let $f \in \mathbb{K}[[\mathbf{x}]] \setminus \{0\}$. We say that f admits a *standard representation with respect to R and $<$* if there exists a r-tuple $(q_1, \cdots, q_r) \in \mathbb{K}[[\mathbf{x}]]^r$ that satisfies the following two conditions:

1. The q_i's are cofactor coefficients for f with respect to R, *i.e.*:

$$f = \sum_{i=1}^{r} q_i s_i.$$

2. There are no cancellations of the least leading terms of the $q_i s_i$'s, *i.e.*:

$$\mathrm{lm}\,(f) = \min_< \{\mathrm{lm}\,(q_i s_i) \,|\, 1 \leq i \leq r, q_i \neq 0\}.$$

Remark 41 It is evident from the definition that any non-zero formal power series f that admits a standard representation with respect to R and $<$ is in the ideal $I(R)$ and has a leading monomial $\mathrm{lm}\,(f)$ which is reducible in the system $\mathfrak{X}_R^{<}$. Indeed, choose arbitrarily among the rules in R that reduce the leading monomial of f for the opposite order of $<$, *i.e.* choose any $i_0 \in \{i \in \{1, \cdots, r\} \,|\, q_i \neq 0, \mathrm{lm}\,(q_i s_i) = \mathrm{lm}\,(f)\}$, then:

$$\mathrm{lm}\,(f) = \mathrm{lm}\,(q_{i_0} s_{i_0}) = \mathrm{lm}\,(q_{i_0}) \cdot \mathrm{lm}\,(s_{i_0}).$$

It is known from HIRONAKA's theorem (see for instance [24, Corollary 2.2]) that R is a standard basis of $I(R)$ for $<$ if, and only if, every non-zero f in $I(R)$ has a standard representation with respect to R and $<$. We shall recover that result in Subsection 3.3, for the case of an order compatible with the degree, using the following proposition.

Proposition 42 *Assume the order $<$ is compatible with the degree. If $f \in \mathbb{K}[[\mathbf{x}]] \setminus \{0\}$ is such that $f \rightarrowtail\!\!\circledast\, 0$, then f admits a standard representation with respect to R and $<$.*

Proof By definition of $f \rightarrowtail\!\!\circledast\, 0$, there exists a sequence $(h_k)_{k\in\mathbb{N}}$ in $\mathbb{K}[[\mathbf{x}]]$ such that $h_0 = f$, for any $k \in \mathbb{N}$, we have $h_k \overset{=}{\longrightarrow} h_{k+1}$ and finally, $\lim_{k\to\infty} h_k = 0$. Let us construct inductively an r-tuple of sequences $\left(q_1^{(k)}, \cdots, q_r^{(k)}\right)_{k\in\mathbb{N}}$ in $\mathbb{K}[[\mathbf{x}]]$.

The base step is to set $q_i^{(0)} := 0$ for any $i \in \{1, \cdots, r\}$. Now, suppose that we have inductively constructed the sequences up to a rank $k \in \mathbb{N}$ such that they satisfy the following condition:

$$f = h_k + \sum_{i=1}^{r} q_i^{(k)} s_i.$$

If $h_k = h_{k+1}$, set $q_i^{(k+1)} := q_i^{(k)}$ for all $i \in \{1, \cdots, r\}$. Otherwise, since $h_k \overset{=}{\longrightarrow} h_{k+1}$, we necessarily have $h_k \longrightarrow h_{k+1}$, which means, by definition, there exist $M_k \in \mathrm{supp}\,(h_k)$, an index $i_k \in \{1, \cdots, r\}$ and a monomial $m_k \in [\mathbf{x}]$ such that $M_k = m_k \cdot \mathrm{lm}\,(s_{i_k})$ and:

$$h_{k+1} = h_k - \frac{\langle h_k \,|\, M_k \rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k}.$$

We then set $q_i^{(k+1)} := q_i^{(k)}$ for all $i \in \{1, \cdots, r\} \setminus \{i_k\}$ and:

$$q_{i_k}^{(k+1)} := q_{i_k}^{(k)} + \frac{\langle h_k \,|\, M_k \rangle}{\mathrm{lc}\,(s_{i_k})} m_k.$$

Note how we then have:

$$\begin{aligned}
h_{k+1} + \sum_{i=1}^{r} q_i^{(k+1)} s_i &= h_{k+1} + \left(q_{i_k}^{(k)} + \frac{\langle h_k \,|\, M_k \rangle}{\mathrm{lc}\,(s_{i_k})} m_k\right) s_{i_k} + \sum_{\substack{i=1 \\ i \neq i_k}}^{r} q_i^{(k)} s_i \\
&= \left(h_{k+1} + \frac{\langle h_k \,|\, M_k \rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k}\right) + \sum_{i=1}^{r} q_i^{(k)} s_i \\
&= h_k + \sum_{i=1}^{r} q_i^{(k)} s_i \\
&= f.
\end{aligned}$$

Hence the induction hypothesis is verified for $k+1$ and we can continue. Repeating this process *ad infinitum* we obtain r sequences $\left(q_i^{(k)}\right)_{k\in\mathbb{N}}$ which turn out to be Cauchy. Indeed, we notice that, by compatibility with the degree, the sequence $(\deg(m_k))_{k\in\mathbb{N}}$ is always eventually bigger than any positive integer because so is the sequence $(\deg(M_k))_{k\in\mathbb{N}}$ of reduced monomials in the Cauchy rewriting sequence $(h_k)_{k\in\mathbb{N}}$. Since $\mathbb{K}[[\mathbf{x}]]$ is a complete metric space, there exists, for each $i \in \{1, \cdots, r\}$, $q_i^{(\infty)} \in \mathbb{K}[[\mathbf{x}]]$ such that $\lim_{k\to\infty} q_i^{(k)} = q_i^{(\infty)}$. The induction hypothesis allows us to write:

$$f = \lim_{k\to\infty}\left(h_k + \sum_{i=1}^{r} q_i^{(k)} s_i\right) = 0 + \sum_{i=1}^{r} q_i^{(\infty)} s_i.$$

Because f rewrites into 0 via chains, there exists a unique $k \in \mathbb{N}$ such that $M_k = \mathrm{lm}\,(f)$. It follows that $q_{i_k}^{(\infty)} \neq 0$ and $\mathrm{lm}\left(q_{i_k}^{(\infty)} s_{i_k}\right) = M_k = \mathrm{lm}\,(f)$. Hence, the $q_i^{(\infty)}$'s do exhibit a standard representation for f with respect to R and $<$. $\square$

3.3 The congruence relation modulo the ideal and the characterisations of standard bases

In this subsection, we will show that the congruence relation modulo the ideal $I(R)$ is exactly the equivalence relation generated by $\succ\!\!\!\multimap$, when the order $<$ is assumed to be compatible with the degree. Under this assumption, we will also prove that this latter equivalence relation is the same as the equivalence relation generated by $\multimap$.

Let $\equiv_{I(R)}$ be the congruence relation modulo the ideal $I(R)$, that is to say, we have $f \equiv_{I(R)} g$ if and only if $f - g \in I(R)$. Denote by $\overset{\star}{\circ\!\!-\!\!\diamond\!\!\!\multimap}$ (resp. $\overset{\star}{\circ\!\!-\!\!\!\multimap}$) the equivalence relation generated by $\succ\!\!\!\multimap$ (resp. $\multimap$): it is the transitive closure of the symmetric closure, denoted $\circ\!\!-\!\!\diamond\!\!\!\multimap$ (resp. $\circ\!\!-\!\!\!\multimap$), of the relation $\succ\!\!\!\multimap$ (resp. $\multimap$).

We are first going to prove a few lemmas.

Lemma 43 ([17, Proposition 4.2.2]) *For all $f, g \in \mathbb{K}[[\mathbf{x}]]$, if $f \multimap g$, then $f \equiv_{I(R)} g$.*

This lemma is true only because, as we mentioned before, ideals of (commutative) formal power series are topologically closed for the topology τ_δ.

Lemma 44 *Assume the order $<$ is compatible with the degree. Then, for all $q \in \mathbb{K}[[\mathbf{x}]]$ and all $i \in \{1, \cdots, r\}$, we have $qs_i \succ\!\!\!\multimap 0$.*

Proof Because the order is compatible with the degree and thus of order type ω, we can write the support of q as a (potentially infinite) sequence $(m_k)_{k\in\mathbb{N}}$ of monomials that is strictly increasing for $<$. If that sequence is finite, then it is clear that, after rewriting each monomial of the form $M_k := m_k \cdot \mathrm{lm}\,(s_i) \in \mathrm{supp}\,(qs_i)$ using the rule of index i, everything will cancel out and we would have $qs_i \xrightarrow{\star} 0$. It then suffices to decompose that relation in one-step reduction relations and complete to infinity with zeros to end up with a sequence which satisfies $qs_i \succ\!\!\!\multimap 0$. Otherwise, if the support is infinite, we define $h_0 = qs_i$ and, for each $k \in \mathbb{N}$:

$$h_{k+1} := h_k - \frac{\mathrm{lc}\,(h_k)}{\mathrm{lc}\,(s_i)} m_k s_i.$$

It is straightforward to see that, for every $k \in \mathbb{N}$, we have $h_k \longrightarrow h_{k+1}$, because we rewrite the leading monomial of h_k, which is of the form $M_k := m_k \cdot \mathrm{lm}\,(s_i)$, with rule of index i. Hence, by Lemma 36, the sequence $(h_k)_{k\in\mathbb{N}}$ converges. Moreover, it is clear that its limit is 0 because, the sequence $(m_k)_{k\in\mathbb{N}}$ being strictly increasing for $<$ and the order $<$ being assumed to be compatible with the degree, it follows that the sequence $(\deg(M_k))_{k\in\mathbb{N}}$ is eventually bigger than any positive integer and the sequence $(M_k)_{k\in\mathbb{N}}$ is exactly the sequence of leading monomials of the sequence $(h_k)_{k\in\mathbb{N}}$. □

Lemma 45 (Translation lemma) *Assume the order $<$ is compatible with the degree. For all formal power series $f, g, h \in \mathbb{K}[[\mathbf{x}]]$, if $f - g \mathrel{\succ\!\!\!\circledast} h$, then there exist $f', g' \in \mathbb{K}[[\mathbf{x}]]$ such that $h = f' - g'$, $f \mathrel{\succ\!\!\!\circledast} f'$ and $g \mathrel{\succ\!\!\!\circledast} g'$.*

Proof By definition, we have a sequence $(h_k)_{k\in\mathbb{N}}$ such that $h_0 = f - g$, $\lim_{k\to\infty} h_k = h$ and, for all $k \in \mathbb{N}$, $h_k \xrightarrow{=} h_{k+1}$. Construct inductively two sequences $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$. First, set $f_0 := f$ and $g_0 := g$. Assume the sequences are constructed up to a rank $k \in \mathbb{N}$ such that $h_k = f_k - g_k$. If we have $h_k = h_{k+1}$, we set $f_{k+1} := f_k$ and $g_{k+1} := g_k$. Otherwise, if $h_k \neq h_{k+1}$, we necessarily have $h_k \longrightarrow h_{k+1}$, which means by definition, that there exist $M_k \in \mathrm{supp}\,(h_k)$, an index $i_k \in \{1, \cdots, r\}$ and a monomial $m_k \in [\mathbf{x}]$ such that $M_k = m_k \cdot \mathrm{lm}\,(s_{i_k})$ and:

$$h_{k+1} = h_k - \frac{\langle h_k \mid M_k\rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k}.$$

We then set:

$$f_{k+1} := f_k - \frac{\langle f_k \mid M_k\rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k} \qquad \text{and} \qquad g_{k+1} := g_k - \frac{\langle g_k \mid M_k\rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k}.$$

Note how, then, by induction, we have:

$$\begin{aligned}
h_{k+1} &= h_k - \frac{\langle h_k \mid M_k\rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k} \\
&= (f_k - g_k) - \frac{\langle f_k - g_k \mid M_k\rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k} \\
&= \left(f_k - \frac{\langle f_k \mid M_k\rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k}\right) - \left(g_k - \frac{\langle g_k \mid M_k\rangle}{\mathrm{lc}\,(s_{i_k})} m_k s_{i_k}\right) \\
&= f_{k+1} - g_{k+1}.
\end{aligned}$$

If $M_k \notin \mathrm{supp}\,(f_k)$, then $f_k = f_{k+1}$; otherwise, $f_k \longrightarrow f_{k+1}$ by definition. Hence, $f_k \xrightarrow{=} f_{k+1}$, for any $k \in \mathbb{N}$. Same reasoning applies to the sequence $(g_k)_{k\in\mathbb{N}}$. Since the order is compatible with the degree, we can use Lemma 36 to conclude that the sequences $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ have limits; denote them by f' and g' respectively. It is then clear that $f \mathrel{\succ\!\!\!\circledast} f'$ and $g \mathrel{\succ\!\!\!\circledast} g'$. Finally, we conclude with:

$$h = \lim_{k\to\infty} h_k = \lim_{k\to\infty} (f_k - g_k) = \lim_{k\to\infty} f_k - \lim_{k\to\infty} g_k = f' - g'.$$

□

We will actually require only a particular case of that lemma that we state thereafter without the straightforward proof.

Corollary 46 *Assume the order $<$ is compatible with the degree. Let $f, g \in \mathbb{K}[[\mathbf{x}]]$. If we have $f - g \mathrel{\succ\!\!\!\circledast} 0$, then there exists $h \in \mathbb{K}[[\mathbf{x}]]$ such that $f \mathrel{\succ\!\!\!\circledast} h \mathrel{\circledast\!\!\!\prec} g$.*

Lemma 47 *Assume the order $<$ is compatible with the degree. Then $\equiv_{I(R)}$ is a subrelation of $\overset{\star}{\overset{\diamond}{\leftrightarrow}}$.*

Proof First notice how for any subset R' such that $R \subseteq R' \subseteq \mathbb{K}[[\mathbf{x}]] \setminus \{0\}$, the relation $\rightarrowtail$ of the system $\mathfrak{X}_R^{<}$ is a subrelation of the topological rewriting with chains relation with respect to R' and $<$. Now, to get the proposition, it suffices to prove that for all $r \in \mathbb{N}$, if there exist $(q_1, \cdots, q_r) \in \mathbb{K}[[\mathbf{x}]]^r$ such that $f - g = \sum_{i=1}^{r} q_i s_i$, then we have $f \overset{\star}{\overset{\diamond}{\leftrightarrow}} g$. Let us now prove this by induction on the number r of rules in R. The base step is when $r = 0$ which means that $f = g$, and since $\overset{\star}{\overset{\diamond}{\leftrightarrow}}$ is an equivalence relation it is in particular reflexive; the result follows. By induction, let $r \in \mathbb{N}$ satisfy the induction hypothesis and consider $s \in \mathbb{K}[[\mathbf{x}]] \setminus (R \cup \{0\})$, $f, g \in \mathbb{K}[[\mathbf{x}]]$ and $(q, q_1, \cdots, q_r) \in \mathbb{K}[[\mathbf{x}]]^{r+1}$ such that:

$$g - f = qs + \sum_{i=1}^{r} q_i s_i.$$

Let $\underset{R_s}{\rightarrowtail}$ be the topological rewriting with chains relation with respect to $R \cup \{s\}$ and $<$ and denote by $\underset{R_s}{\overset{\star}{\overset{\diamond}{\leftrightarrow}}}$ the equivalence relation generated by it. By Lemma 44, we have $qs \underset{R_s}{\rightarrowtail} 0$. Applying Corollary 46 to $f := f$ and $g := f + qs$, we obtain a formal power series $h \in \mathbb{K}[[\mathbf{x}]]$ such that $f \underset{R_s}{\rightarrowtail} h$ and $f + qs \underset{R_s}{\rightarrowtail} h$. It follows that $f \underset{R_s}{\overset{\star}{\overset{\diamond}{\leftrightarrow}}} f + qs$. Now, notice that, by definition, we have $g - (f + qs) = \sum_{i=1}^{r} q_i s_i$. Apply the induction hypothesis to obtain $g \overset{\star}{\overset{\diamond}{\leftrightarrow}} f + qs$. But, $\rightarrowtail$ is a subrelation of $\underset{R_s}{\rightarrowtail}$ (and hence, $\overset{\star}{\overset{\diamond}{\leftrightarrow}}$ is also a subrelation of $\underset{R_s}{\overset{\star}{\overset{\diamond}{\leftrightarrow}}}$), so $g \underset{R_s}{\overset{\star}{\overset{\diamond}{\leftrightarrow}}} f + qs \underset{R_s}{\overset{\star}{\overset{\diamond}{\leftrightarrow}}} f$, therefore, $f \underset{R_s}{\overset{\star}{\overset{\diamond}{\leftrightarrow}}} g$, which is the desired result. □

We can finally prove that the congruence relation is characterised by the equivalence relations generated by the topological rewriting relations.

Theorem 48 *Assume the order $<$ is compatible with the degree. Then $\equiv_{I(R)}$, $\overset{\star}{\leftrightarrow}$ and $\overset{\star}{\overset{\diamond}{\leftrightarrow}}$ are equal.*

Proof By Lemma 47, $\equiv_{I(R)}$ is a subrelation of $\overset{\star}{\overset{\diamond}{\leftrightarrow}}$. Now, since $\rightarrowtail$ is a subrelation of $\to$, it follows that $\overset{\star}{\overset{\diamond}{\leftrightarrow}}$ is also a subrelation of $\overset{\star}{\leftrightarrow}$. Finally, let us show that $\overset{\star}{\leftrightarrow}$ is a subrelation of $\equiv_{I(R)}$, which will prove the theorem. Let $f, g \in \mathbb{K}[[\mathbf{x}]]$ such that $f \overset{\star}{\leftrightarrow} g$. Decompose this into a finite sequence:

$$f \leftrightarrow h_1 \leftrightarrow h_2 \leftrightarrow \cdots \leftrightarrow h_\ell \leftrightarrow g.$$

Using Lemma 43, it follows that, for any $h, h' \in \mathbb{K}[[\mathbf{x}]]$, $h \leftrightarrow h'$ implies $h - h' \in I(R)$. Thus, by induction on the finite sequence above, we have:

$$f - g = (f - h_1) + (h_1 - h_2) + \cdots + (h_\ell - g) \in I(R).$$

From which we deduce $f \equiv_{I(R)} g$. □

The following theorem is where the main contributions of this paper can be found.

Theorem 49 *Assume the order $<$ is compatible with the degree. Then the following statements are equivalent:*

1. *The system $\mathfrak{X}_R^<$ is finitary topologically confluent.*
2. *For all $f \in I(R)$, we have $f \overset{\circ}{\longrightarrow} 0$.*
3. *For all $f \in I(R) \setminus \{0\}$, f is reducible (i.e. not a normal form).*
4. *The set R is a standard basis of $I(R)$ for $<$.*
5. *For all $f \in I(R)$, we have $f \overset{\circ}{\rightarrowtail} 0$.*
6. *For all $f \in I(R) \setminus \{0\}$, f admits a standard representation with respect to R and $<$.*
7. *For all $f \in \mathbb{K}[[\mathbf{x}]]$, there is a unique $\alpha \in \mathrm{NF}\,(\mathfrak{X}_R^<)$ such that $f = \alpha + \sum_{i=1}^{r} q_i s_i$ for some $(q_1, \cdots, q_r) \in \mathbb{K}[[\mathbf{x}]]^r$.*
8. *The map from $\mathrm{NF}\,(\mathfrak{X}_R^<)$ to the quotient $\mathbb{K}[[\mathbf{x}]]/I(R)$ which associates to any normal form $\alpha \in \mathrm{NF}\,(\mathfrak{X}_R^<)$ the equivalence class $\alpha + I(R)$ is bijective.*
9. *The system $\mathfrak{X}_R^<$ is infinitary topologically confluent.*

Proof Most results use the compatibility with the degree assumption.

1. $\Rightarrow$ 2.: Assuming finitary topological confluence of the system and thanks to Propositions 35, 38, and 39, we can use Corollary 32 of Theorem 31 to deduce that the normal form property with respect to $\overset{\circ}{\longrightarrow}$ is verified. Now, let $f \in I(R)$. We thus have $f \equiv_{I(R)} 0$. By Theorem 48, we have $f \overset{\star}{\overset{\circ}{\longleftrightarrow}} 0$. By the normal form property, since 0 is obviously a normal form of the system, we have $f \overset{\circ}{\longrightarrow} 0$ since $\overset{\circ}{\longrightarrow}$ is transitive.

2. $\Rightarrow$ 3.: Let $f \in I(R) \setminus \{0\}$. We have by assumption $f \overset{\circ}{\longrightarrow} 0$. By contradiction, assume f is a normal form. Then, by Lemma 23, $f = 0$ which contradicts the definition of f.

3. $\Rightarrow$ 4.: Let $f \in I(R) \setminus \{0\}$. Let us show that $\mathrm{lm}\,(f)$ is reducible. By contradiction, suppose that there is no $i \in \{1, \cdots, r\}$ such that $\mathrm{lm}\,(s_i)$ divides $\mathrm{lm}\,(f)$. By Proposition 38, there exists $\alpha \in \mathrm{NF}\,\left(\mathfrak{X}_R^<\right)$ such that $f \overset{\circ}{\rightarrowtail} \alpha$. Since $\overset{\circ}{\rightarrowtail}$ is a subrelation of $\overset{\circ}{\longrightarrow}$, we get $f \overset{\circ}{\longrightarrow} \alpha$. From Lemma 43, we get $f - \alpha \in I(R)$. But, because $f \in I(R)$, we have $\alpha \in I(R)$. Now, note how if $\alpha = 0$, then $f \overset{\circ}{\rightarrowtail} 0$, and therefore $\mathrm{lm}\,(f)$ is necessarily reducible. Otherwise, we would have $\alpha \in I(R) \setminus \{0\}$ and, by assumption, we would deduce that α is not a normal form; a contradiction.

4. $\Rightarrow$ 5.: Let $f \in I(R)$. Set $f_0 := f$. If $f_0 = 0$, it is trivial to see that $f \overset{\circ}{\rightarrowtail} 0$. Assume thus $f_0 \neq 0$. Then $\mathrm{lm}\,(f_0)$ is reducible. Rewrite it to obtain f_1. We thus have $f_0 \longrightarrow f_1$. If $f_1 = 0$, the result is clear. Otherwise, $f_1 \in I(R) \setminus \{0\}$. Therefore, its leading monomial is once again reducible. Rewrite it to obtain f_2, and go on like this forever or until we reach, for some $k \in \mathbb{N}$, the identity $f_k = 0$. If that is the case, then it is clear that $f \overset{\circ}{\rightarrowtail} 0$. Otherwise, if there is never any $k \in \mathbb{N}$ such that $f_k = 0$ then we obtain an infinite sequence $(f_k)_{k \in \mathbb{N}}$ that satisfies the relations $f_0 = f$ and $f_k \longrightarrow f_{k+1}$ for all $k \in \mathbb{N}$. By Lemma 36, it follows that the sequence converges. Moreover, since we always rewrite the leading monomial and since the order is of type ω, being compatible with the degree, we conclude that the limit is 0 which exactly means that $f \overset{\circ}{\rightarrowtail} 0$.

5. $\Rightarrow$ 6.: This is exactly the content of Lemma 42.

6. $\Rightarrow$ 7.: Let $f \in \mathbb{K}[[\mathbf{x}]]$. If f is a normal form, the existence part of 7. is guaranteed. Otherwise, by Proposition 38, there exists $\alpha \in \mathrm{NF}\,\left(\mathfrak{X}_R^<\right)$ different from f such that $f \overset{\circ}{\rightarrowtail} \alpha$. Now, since $\overset{\circ}{\rightarrowtail}$ is a subrelation of $\overset{\circ}{\longrightarrow}$, we obtain by Lemma 43 that $f - \alpha \in I(R) \setminus \{0\}$. Hence, by assumption, $f - \alpha$ admits a standard representation with respect to R and $<$, which yields the existence of the cofactors $(q_1, \cdots, q_r) \in \mathbb{K}[[\mathbf{x}]]^r$ such that $f - \alpha = \sum_{i=1}^{r} q_i s_i$. Let us now show the uniqueness of $\alpha \in \mathrm{NF}\,\left(\mathfrak{X}_R^<\right)$ satisfying that property. Suppose we have

$\beta \in \mathrm{NF}\left(\mathfrak{X}_R^<\right)$ different from α and $(q'_1, \cdots, q'_r) \in \mathbb{K}[[\mathbf{x}]]^r$ such that $f = \sum_{i=1}^r q'_i s_i + \beta$. Then $\alpha - \beta = \sum_{i=1}^r (q_i - q'_i) s_i \in I(R) \setminus \{0\}$. By assumption, $\alpha - \beta$ admits a standard representation with respect to R and $<$. Then, according to Remark 41, it follows that the leading monomial of $\alpha - \beta$ is reducible. However, this would mean that, among α and β, at least one of them contains a reducible monomial in its support, a contradiction. Hence, $\alpha = \beta$.

7. $\Rightarrow$ 8.: It is a rather straightforward reformulation.

8. $\Rightarrow$ 9.: Let $f, g, h \in \mathbb{K}[[\mathbf{x}]]$ such that $f \mathrel{-\!\!\circledast} g$ and $f \mathrel{-\!\!\circledast} h$. By Proposition 38, there exist normal forms $\alpha, \beta \in \mathrm{NF}\left(\mathfrak{X}_R^<\right)$ such that $g \mathrel{\succ\!\!\circledast} \alpha$ and $h \mathrel{\succ\!\!\circledast} \beta$. As $\mathrel{\succ\!\!\circledast}$ is a subrelation of $\mathrel{-\!\!\circledast}$, we obtain thus that:

$$\alpha \mathrel{\circledast\!\!-} g \mathrel{\circledast\!\!-} f \mathrel{-\!\!\circledast} h \mathrel{-\!\!\circledast} \beta.$$

It is clear from Lemma 43 that $\alpha - \beta = (\alpha - g) + (g - f) + (f - h) + (h - \beta) \in I(R)$. Thus, by assumption, $\alpha = \beta$. Hence, the system is infinitary topologically confluent, since we have:

$$\begin{tikzcd}
 & f \arrow[dl] \arrow[dr] & \\
g \arrow[dr] & & h \arrow[dl] \\
 & \alpha = \beta. &
\end{tikzcd}$$

9. $\Rightarrow$ 1.: This is the content of Remark 28. $\square$

Conclusion

In this paper, we demonstrated why Gröbner bases fail to work for complete local equicharacteristic Noetherian rings and we gave convenient rewriting-theoretic characterisations of standard bases for ideals of formal power series, which enables to compute in those rings. Further work in that field could focus on the establishment of a "topologised" Newman's lemma from the classical case. It would then in particular allow us to recover the analogous statement of the Buchberger criterion for standard bases. Note however how the desired end result about that S-series criterion has already been proven in [25], so it would once again be about having purely rewriting-theoretic proofs of the already known results. On other perspectives, there is still the open problem of the *chains conjecture*: given an admissible monomial order $<$ compatible with the degree and R a finite set of rewrite rules for formal power series, are the relations $\mathrel{-\!\!\circledast}$ and $\mathrel{\succ\!\!\circledast}$ from the topological rewriting system $\mathfrak{X}_R^<$ necessarily equal? This is in general not true for R infinite. If the conjecture turns out to be true, then the topological rewriting relation we work with in this paper would be the analogous as the "strongly convergent transfinite reduction relation" of infinitary Σ/λ-term rewriting. This would mean that abstract topological rewriting theory is indeed a theory encompassing both rewriting on formal power series and infinitary term rewriting in computer science.

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